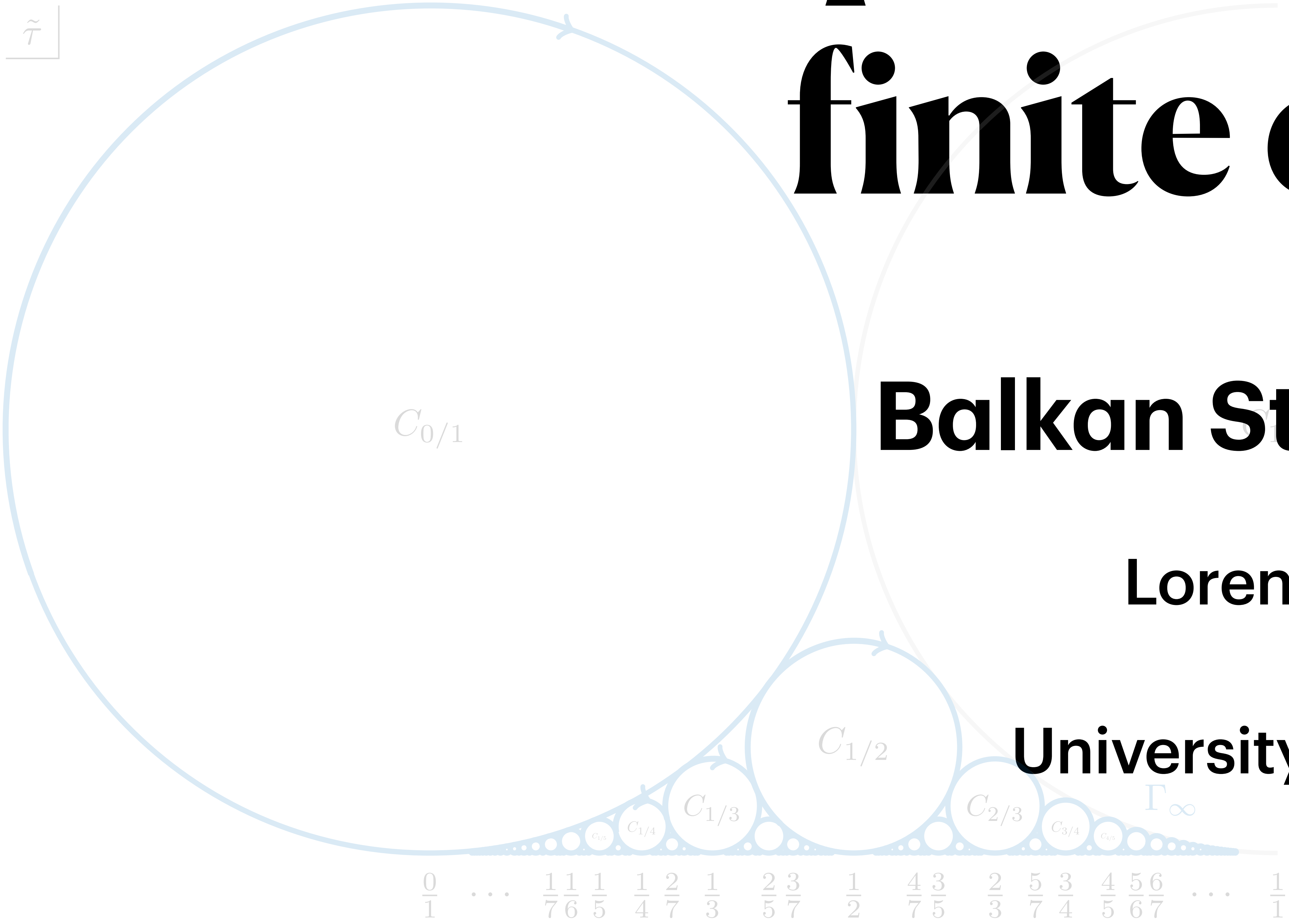
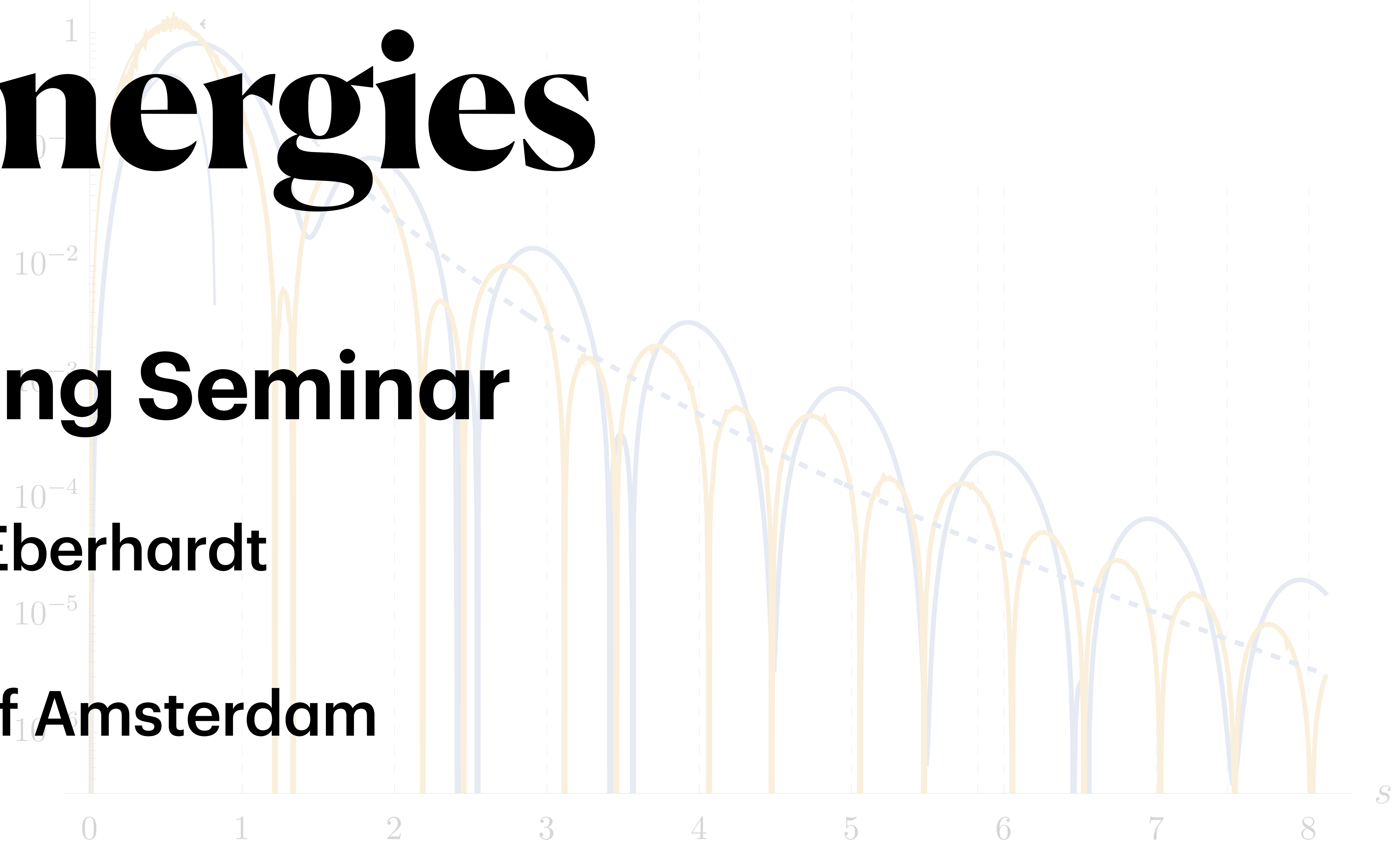
One-loop string amplitudes at finite energies





Balkan String Seminar

Lorenz Eberhardt

University of Amsterdam

Based on work in collaboration with Marco M. Baccianti, Jeevan Chandra, Tom Hartman, Sebastian Mizera
[2208.12233, 2302.12733, 2501.13827, 2507.22105, 2601.09707]

UNIVERSITEIT VAN AMSTERDAM

Surprisingly little is known about higher loop string amplitudes

Virasoro-Shapiro amplitude

$$\mathscr{A}(s,t) = t_8\tilde{t}_8 \ \times \ \frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}$$

Veneziano ’68,
Virasoro ’69,
Shapiro ‘69

Polarization structure

No explicit formulas for higher loops…

Virasoro-Shapiro amplitude

$$\mathscr{A}(s,t) = t_8\tilde{t}_8 \times \boxed{\frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}}$$

Veneziano ’68,
Virasoro ’69,
Shapiro ‘69

Polarization structure

No explicit formulas for higher loops...

Amazing properties:

- Manifestly crossing symmetric (dual resonance)
- Regge bounded
- UV soft
- Unitary

Today I'll explain a general method to evaluate one-loop string amplitudes

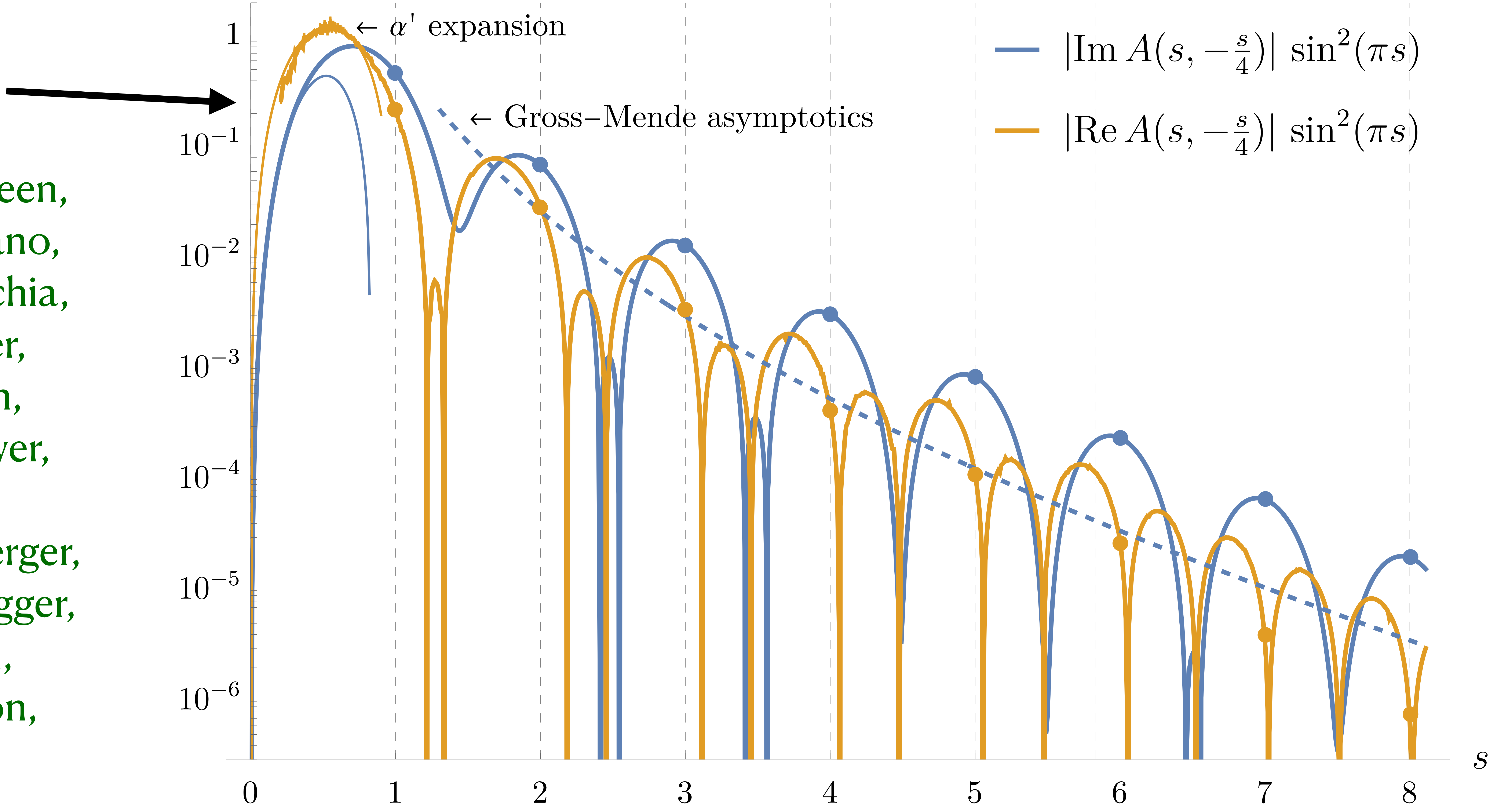
What physical properties do they exhibit?

Type II one-loop amplitude in 10d @ 60 degrees

Field theory

+ α' corrections

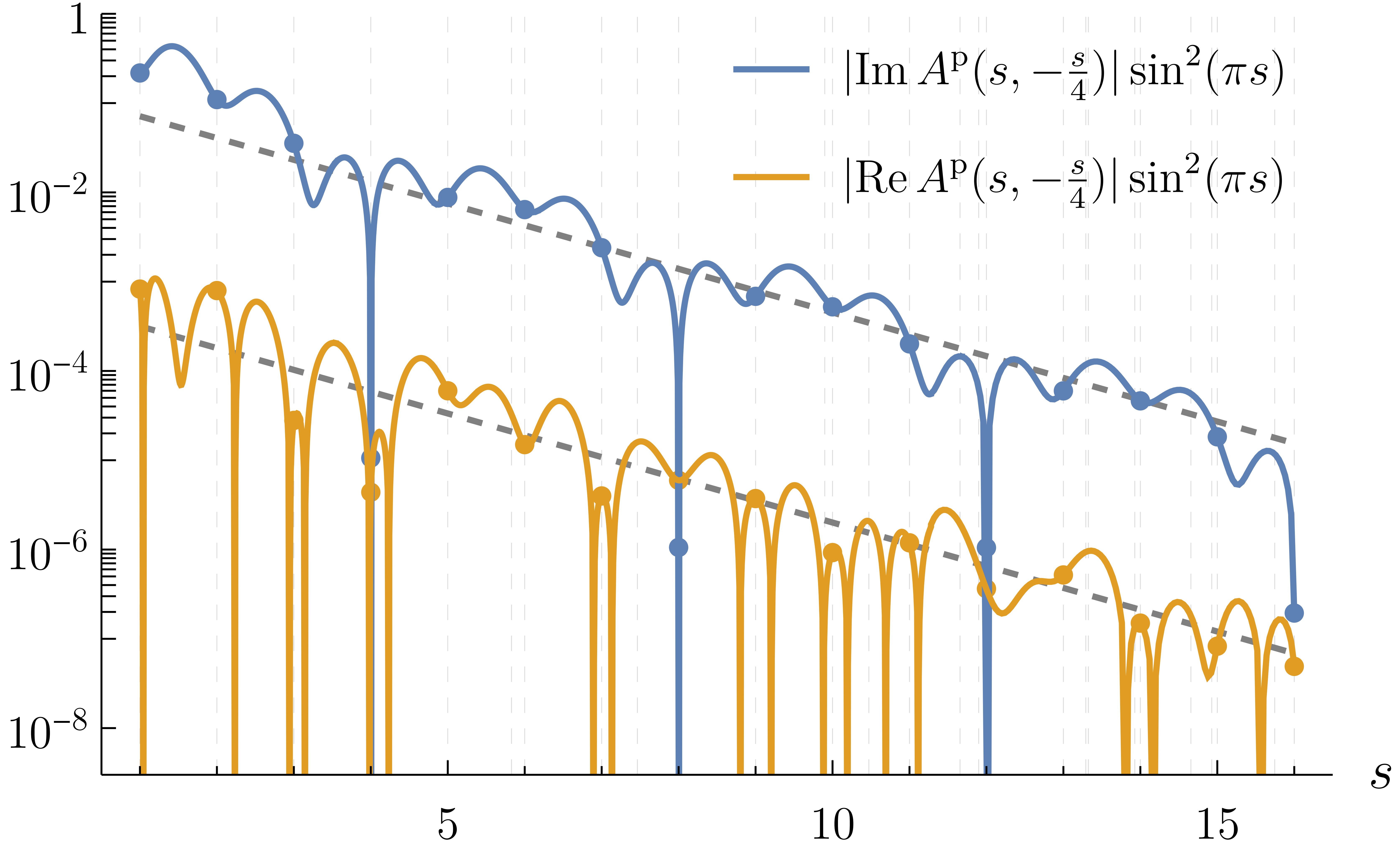
[enormous literature: Green, Schwarz, Gross, Veneziano, Amati, Ciafaloni, Di Vecchia, Koba, Nielsen, D'Hoker, Phong, Martinec, Bern, Dixon, Polyakov, Kosower, Vanhove, Schlotterer, Mafra, Stieberger, Brown, Broedel, Hohenegger, Kleinschmidt, Gerken, Roiban, Lipstein, Mason, Monteiro, Claasen, Doroudiani...]



$\alpha' = 4$

Baccianti, LE, Mizera '25

Type I one-loop amplitude in 10d @ 60 degrees



LE, Mizera ‘23

Outline

I. Defining the amplitude

II. Rademacher evaluation

III. High-energy limit

I. Defining the amplitude

Higher loop amplitudes

Textbook definition:

$$A_{g,n}(p_1, \ldots, p_n) = \int_{\mathcal{M}_{g,n}} (\text{CFT correlation function})$$

Moduli space of Riemann surfaces

Integral does not converge due to expected physical singularities in the amplitude: massive poles & branch cuts

A definition consistent with causality

We treat the worldsheet as Euclidean, but spacetime is Lorentzian!

$$A_{g,n}(p_1, \ldots, p_n) = \int_{\Gamma\subset\mathscr{M}^{\mathbb{C}}_{g,n}} \text{(CFT correlation function)}$$

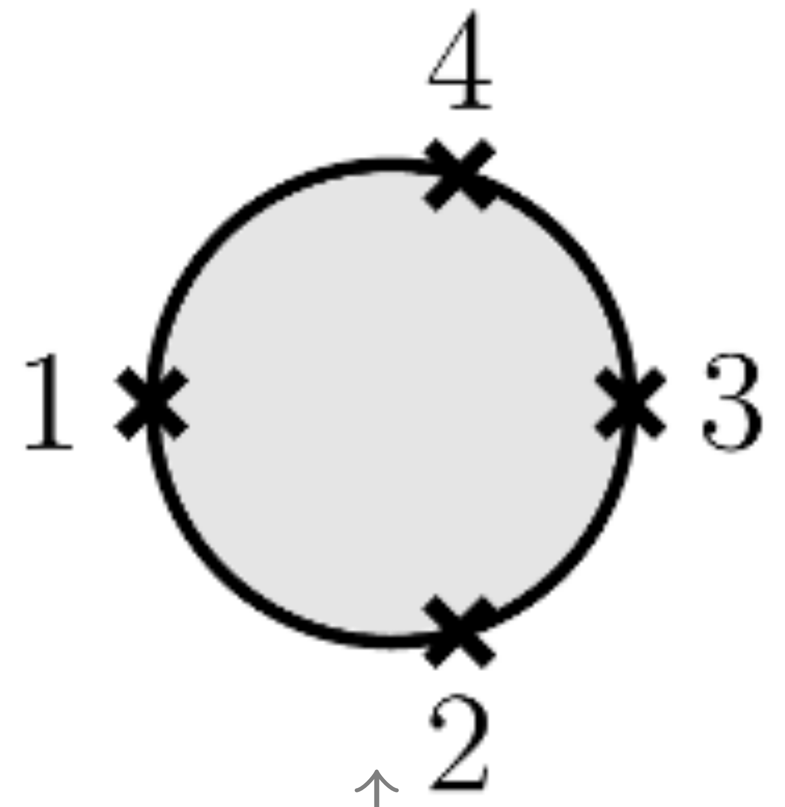
Contour consistent
with the Lorentzian spacetime

Describes complex metrics
on the worldsheet

Modify contour of integration near a degeneration of the worldsheet: stringy $i\epsilon$ prescription

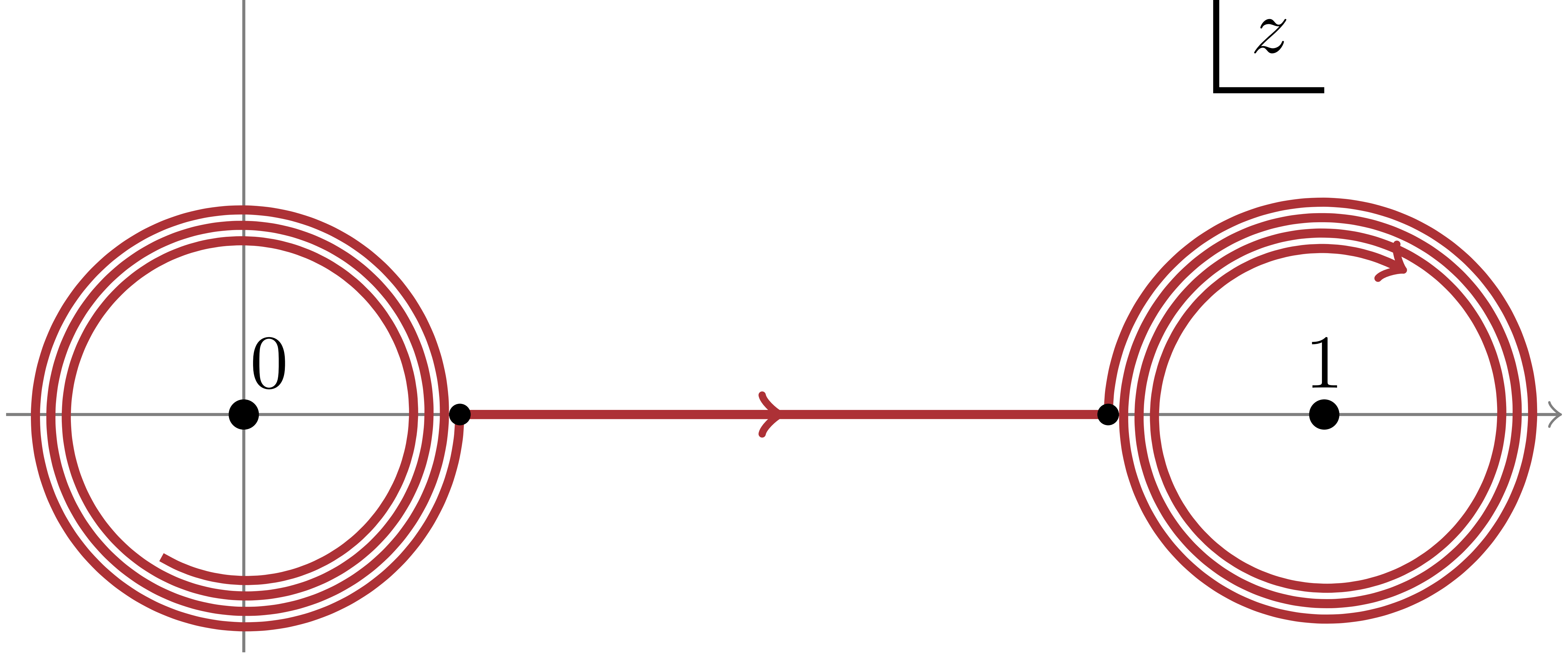
Witten '13

The open tree-level amplitude



$$= \frac{1}{t}\int_{\Gamma} \mathrm{d}z\; z^{-s-1}(1-z)^{-t}$$

$\alpha' = 1$

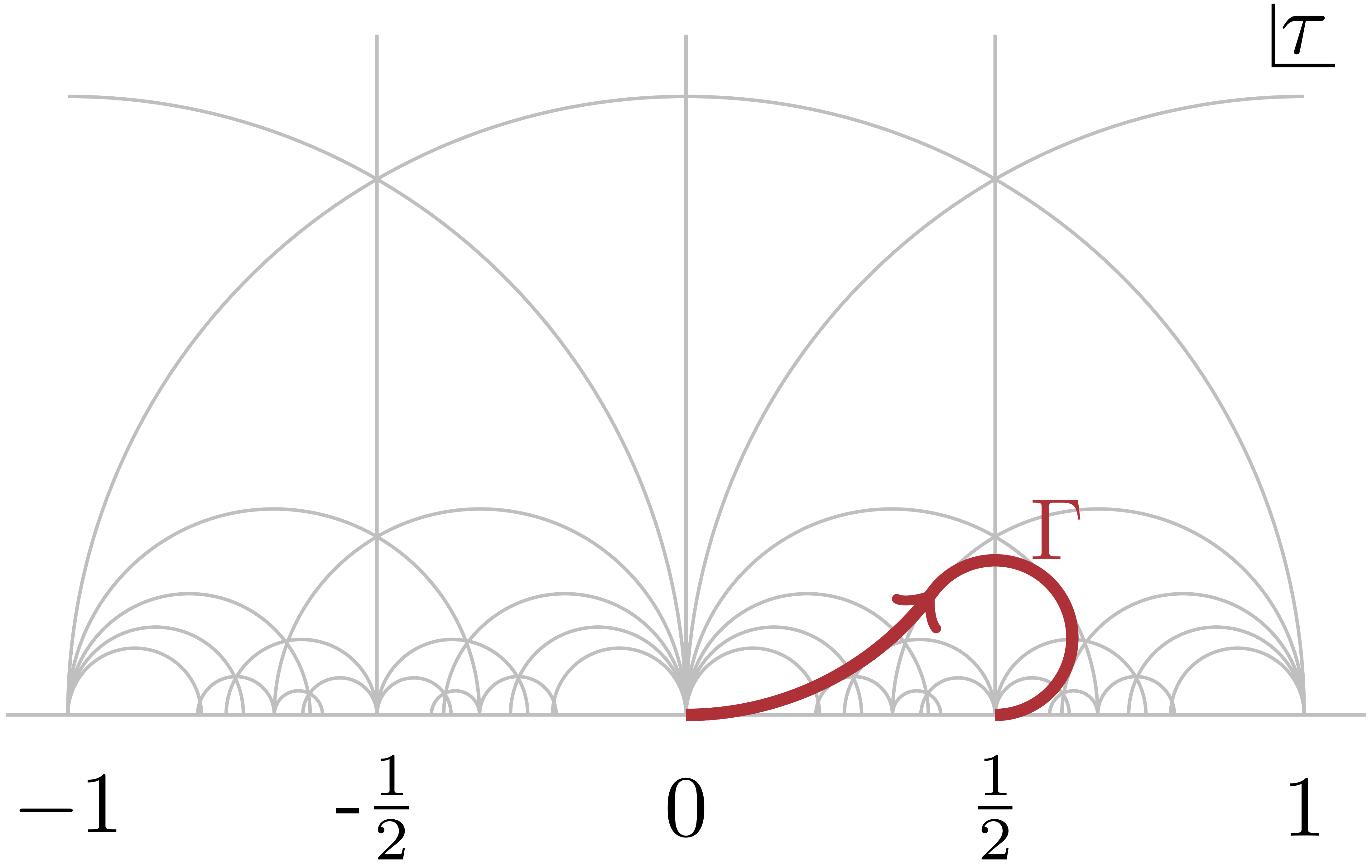


$z = \mathrm{e}^{-\tau}$

Euclidean
Schwinger
parameter

The open one-loop planar amplitude

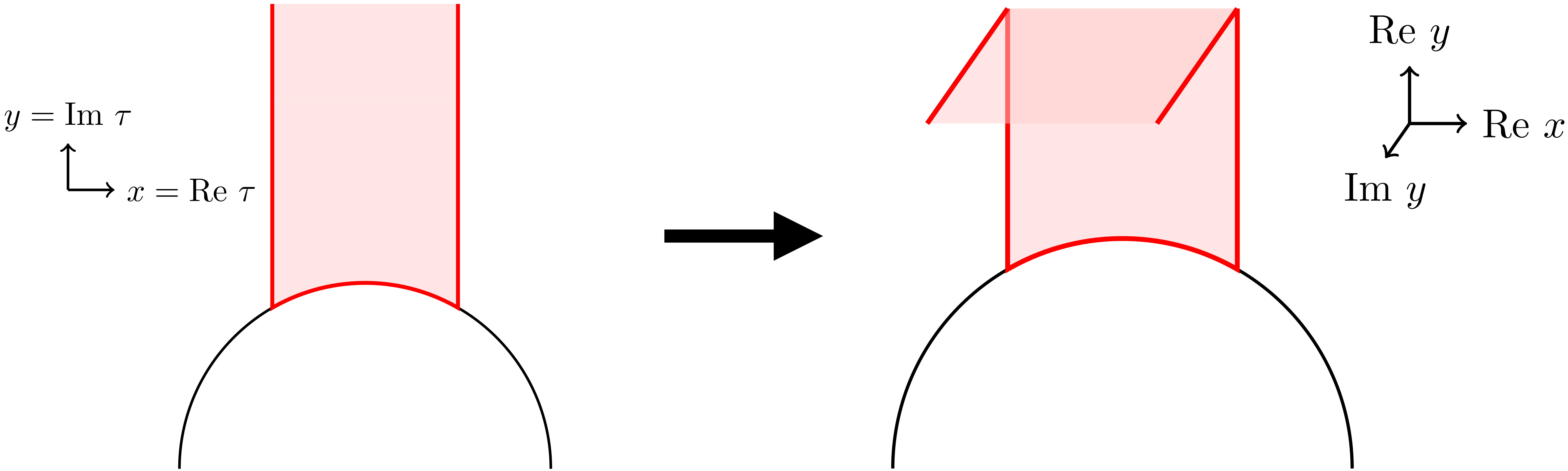
$$A(s,t) = -\,i \int_{\Gamma \subset \mathscr{M}_{1,4}^{\mathrm{open},\mathbb{C}}} \mathrm{d}\tau\, \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{43},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)} \right)^{-s} \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{41},\tau)}{\vartheta_1(z_{32},\tau)\vartheta_1(z_{42},\tau)} \right)^{-t}$$



The closed one-loop amplitude

Green, Schwarz '82

$$A = \int_\Gamma \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^5} \int_{\mathbb{T}^2} \prod_{j=1}^{3} \mathrm{d}^2 z_j \prod_{1\leq j<i\leq 4} |\,\vartheta_1(z_{ij}\,|\,\tau)\,|^{-2s_{ij}}\, \mathrm{e}^{\frac{2\pi s_{ij}(\mathrm{Im} z_{ij})^2}{\mathrm{Im}\tau}}$$



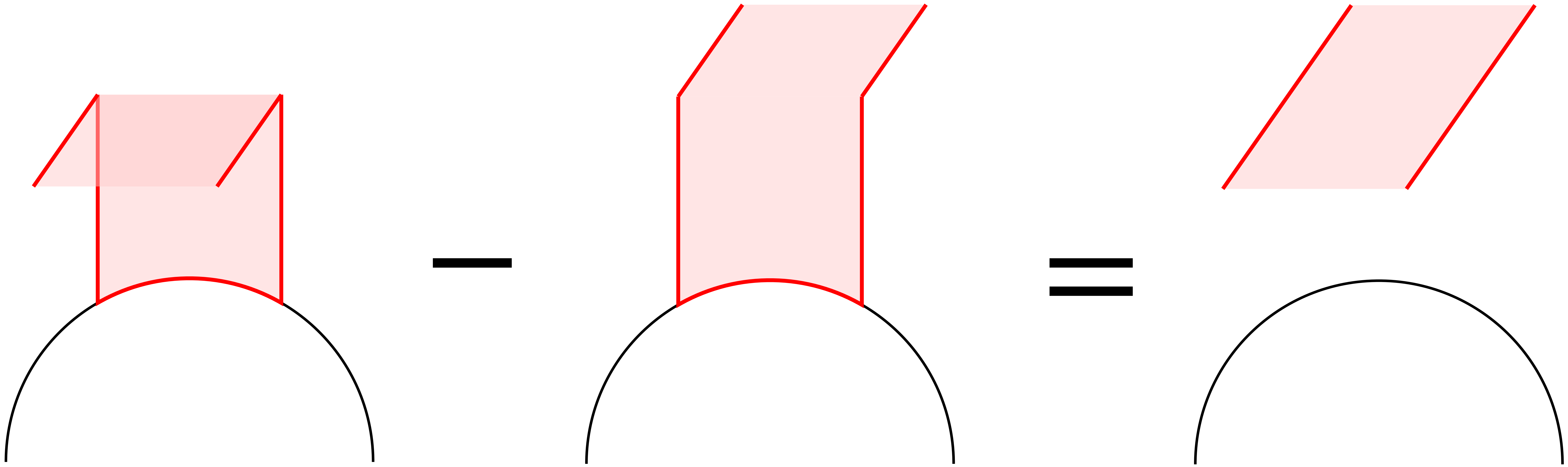
II. Evaluating the amplitude

Contour still not very useful

- Convergent representation of the amplitude, but still very bad for e.g. numerical integration
- Integrand highly oscillatory
- Use contour deformations to proceed!

Warm up: imaginary part

- Difference of contours:



- Only need to know the local behavior of the integrand near the cusp $\mathrm{Im}\,\tau \to \infty$

Toy example

Bosonic string partition function: $Z = \int_{\Gamma} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}\,|\,\eta(\tau)^{24}\,|^2}$

$$\begin{aligned} 2i\mathrm{Im}\,Z &= \int_{\Gamma_{\mathrm{Im}}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}\,|\,\eta(\tau)^{24}\,|^2} \\ &= \int_{\Gamma_{\mathrm{Im}}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}} \left(\mathrm{e}^{4\pi \mathrm{Im}\,\tau} + \ldots \right) \\ &= \int_{1-i\infty}^{1+i\infty} \frac{\mathrm{d}y}{y^{14}}\, \mathrm{e}^{4\pi y} = \frac{i\,2^{27}\pi^{14}}{13\,!} \end{aligned}$$

Subleading terms disappear in the limit $\mathrm{Im}\tau \to \infty$

Baikov representation

Sum over massive states

Integral over on-shell phase space

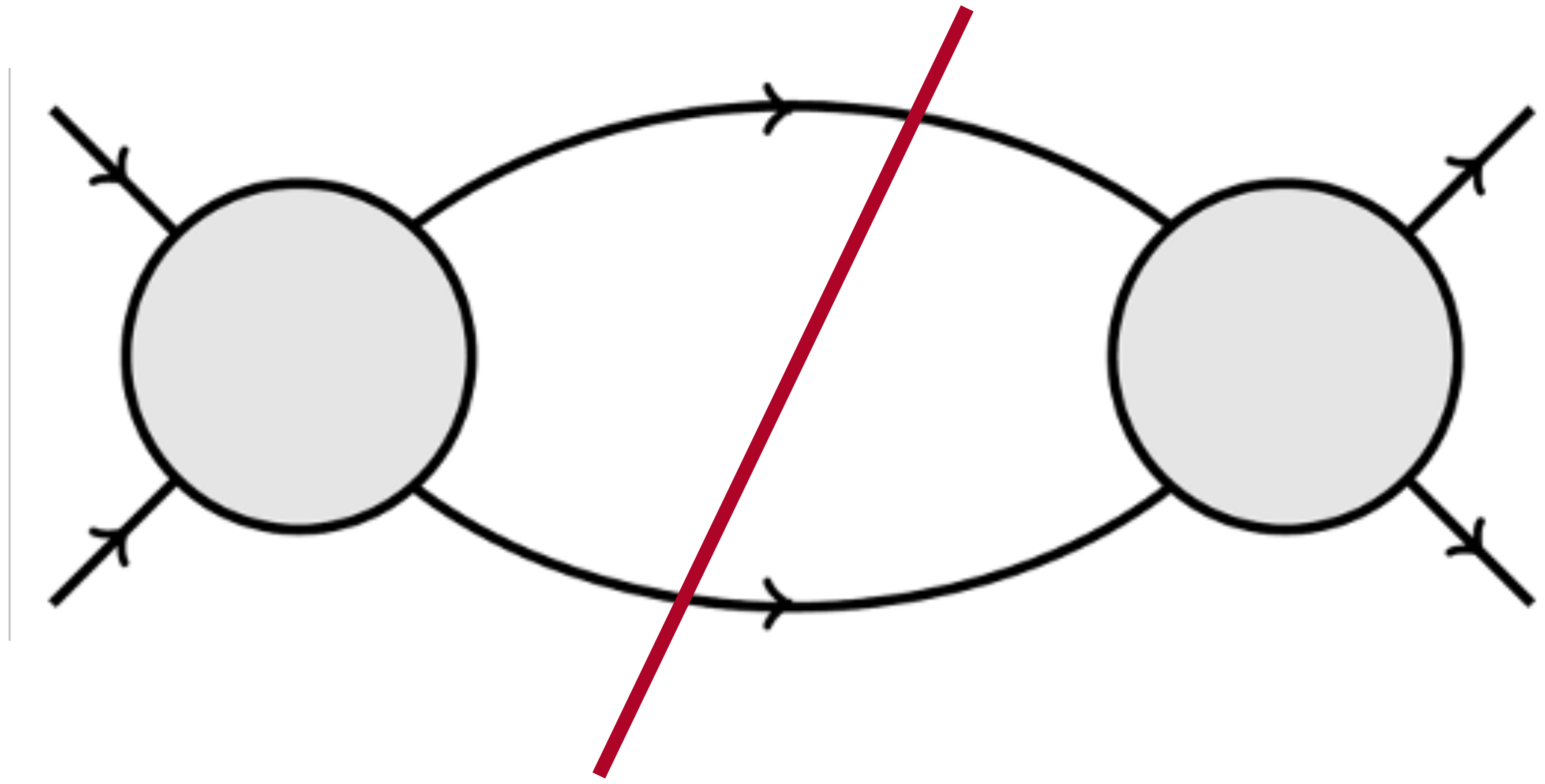
Gram determinant

Sum over polarizations,
Explicit formula via generating function,
Double copies from open string

$$\mathrm{Im}\,A = \frac{16\pi s^4}{15\sqrt{stu}} \sum_{\sqrt{m_\mathrm{D}}+\sqrt{m_\mathrm{U}}\leq\sqrt{s}} \int \mathrm{d}t_\mathrm{L}\,\mathrm{d}t_\mathrm{R}\; P^{\frac{5}{2}}_{m_\mathrm{D},m_\mathrm{U}} Q^{2}_{m_\mathrm{D},m_\mathrm{U}}$$

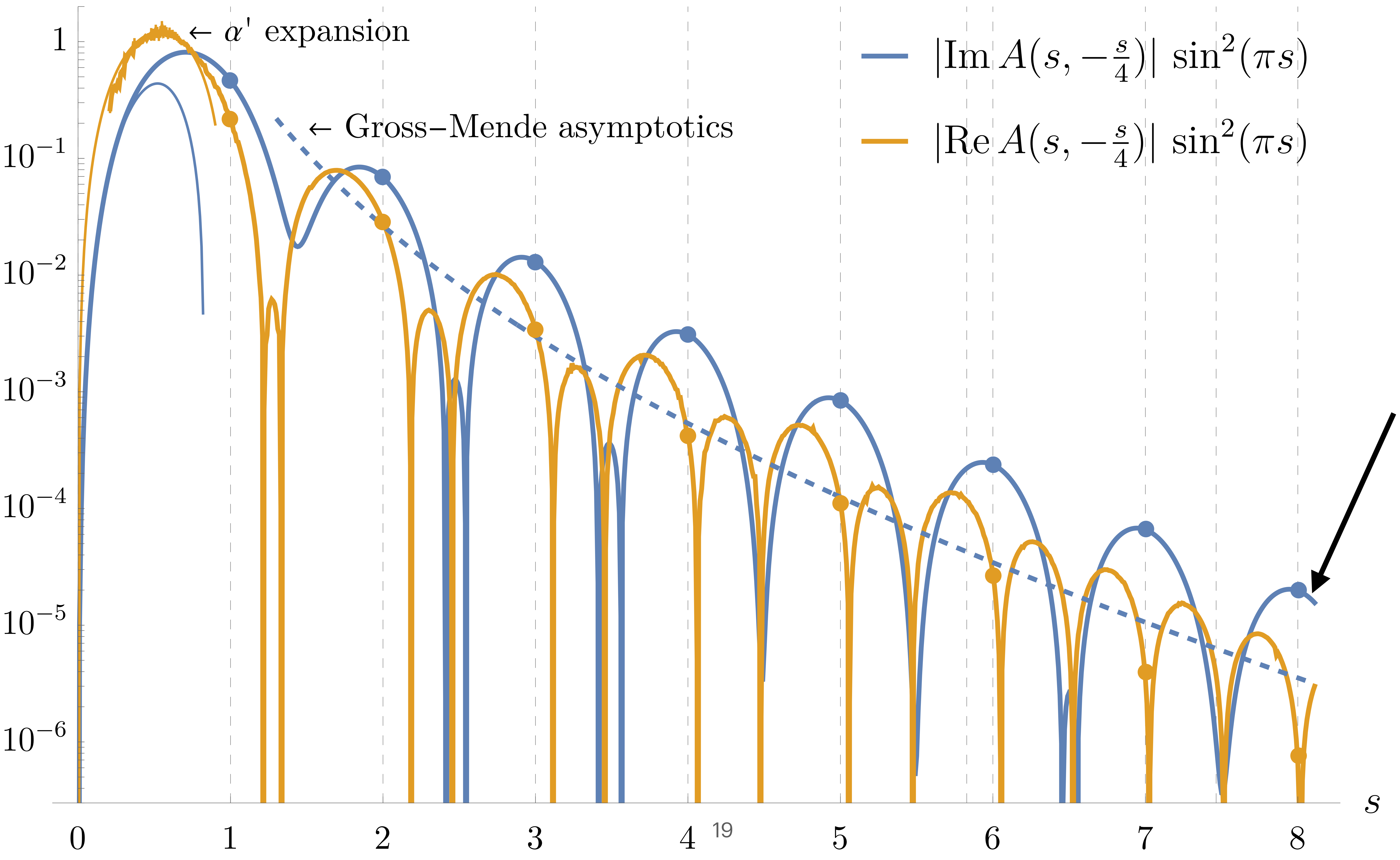
$$\times \frac{\Gamma(-s)\Gamma(-t_\mathrm{L})\Gamma(-u_\mathrm{L})}{\Gamma(1+s)\Gamma(1+t_\mathrm{L})\Gamma(1+u_\mathrm{L})} \times \frac{\Gamma(-s)\Gamma(-t_\mathrm{R})\Gamma(-u_\mathrm{R})}{\Gamma(1+s)\Gamma(1+t_\mathrm{R})\Gamma(1+u_\mathrm{R})}$$

Massive Veneziano amplitudes



LE, Mizera ’22; Banerjee, LE, Mizera ‘24

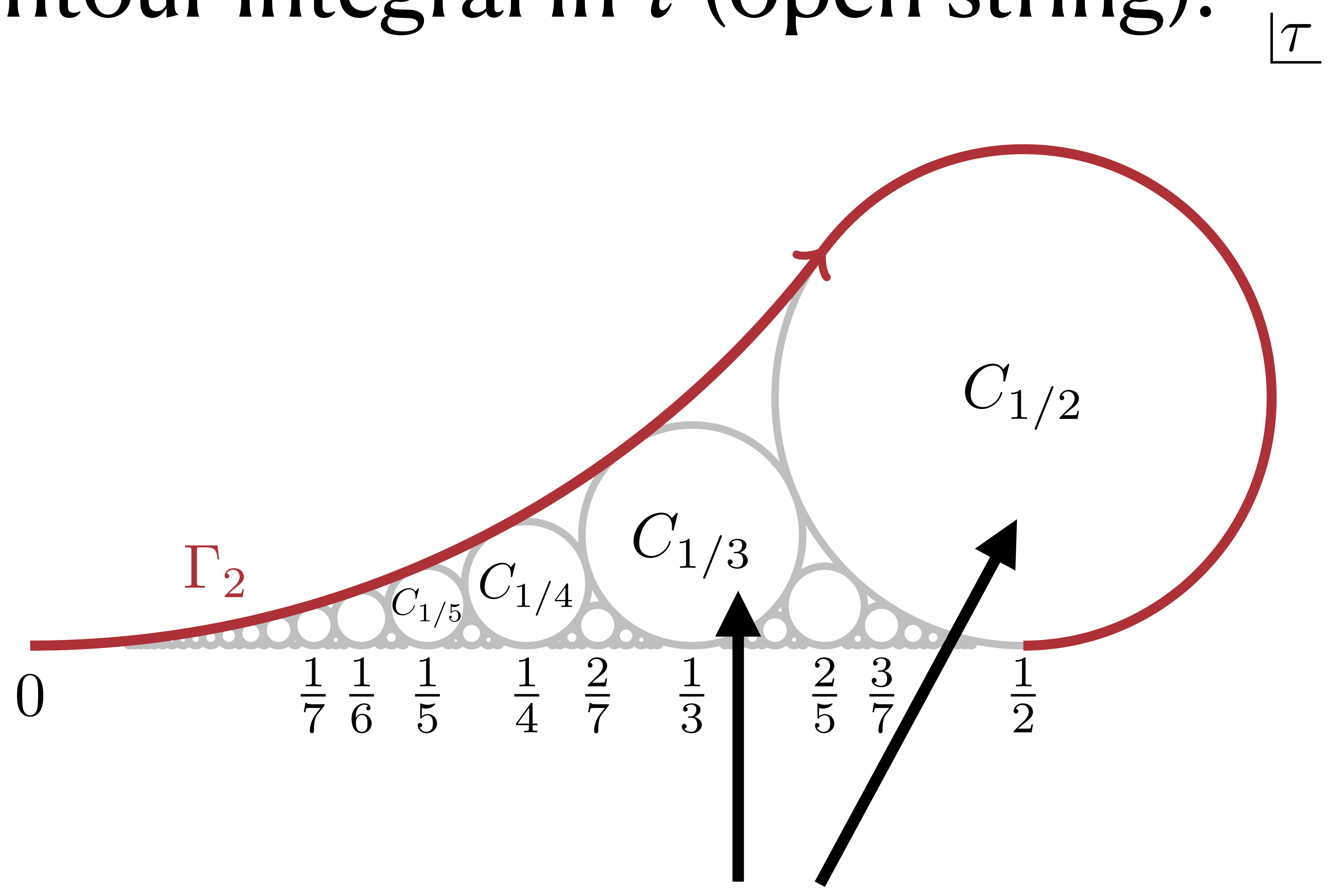
Gives the blue curve



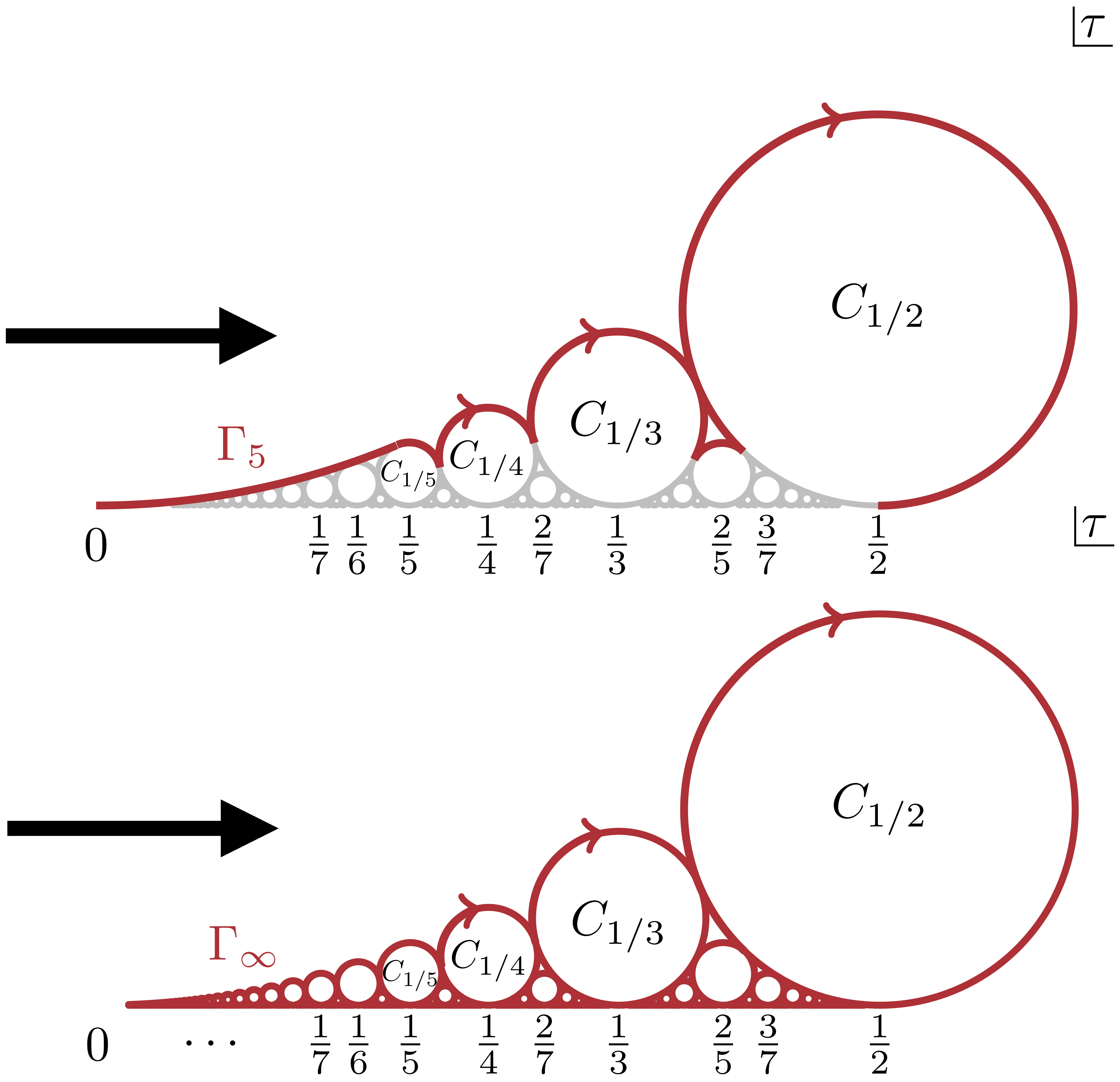
Already exchange of $\mathcal{O}(10^{22})$ states

For the real part: Rademacher expansion

- Idea: deform the contour such that it is only sensitive to the cusps in moduli space
- For a contour integral in τ (open string):



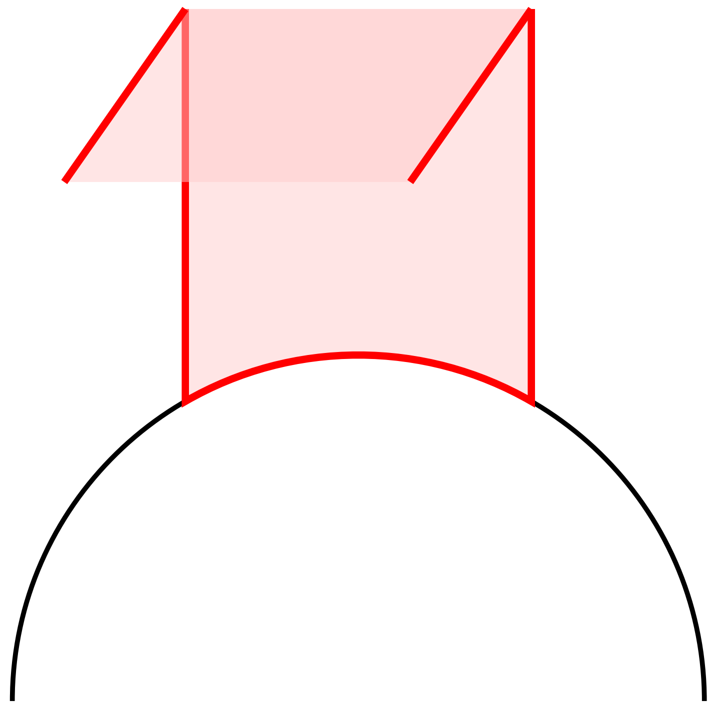
$C_{a/c}$: Ford circles



Rademacher 1943;
For open string: LE, Mizera '23; Manschot, Wang '24

Back to the toy example

$$Z = \int_{\Gamma} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\ \tau)^{14}\,|\,\eta(\tau)^{24}\,|^2}$$



Evaluate it in terms of its behavior near the cusps in the complexified moduli space $(\tau, \tilde{\tau}) \in (\mathbb{H} \times \mathbb{H})/\mathrm{PSL}(2,\mathbb{Z})$ away from the real slice $\tilde{\tau} = -\bar{\tau}$:

Need a two-dimensional version of the Rademacher expansion!

Other approaches: Zagier '81, Lerche, Nilsson, Schellekens, Warner '88; Angelantonj, Florakis, Pioline '10; Korpas, Manschot, Moore, Nidaiev '19; Benjamin, Chang '22

A non-holomorphic Rademacher expansion

$$\operatorname{Re}\int_{\Gamma} \mathrm{d}^2\tau\, f(\tau,\tilde\tau) = \sum_{c=1}^{\infty}\ \sum_{a=0,\,(a,c)=1}^{c-1} \int_{\longrightarrow} \mathrm{d}\tau \int_{C_{a/c}} \mathrm{d}\tilde\tau \left[\frac{1}{12i}\left(\tau-\tilde\tau+\frac{2a}{c}\right)+is(a,c)\right] f(\tau,\tilde\tau)$$

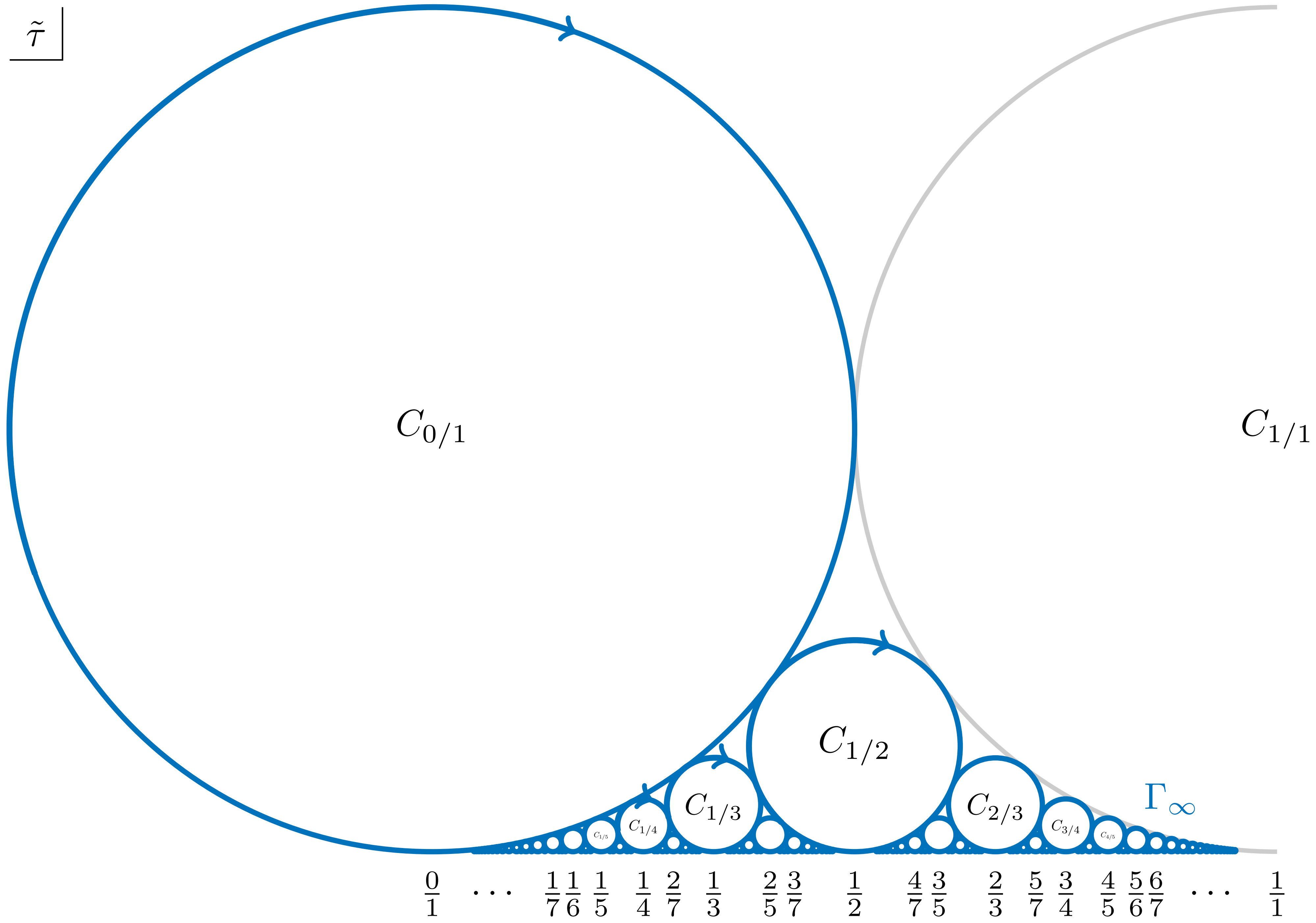
Baccianti, Chandra, Hartman, LE, Mizera ‘25

$$s(a,c) = \sum_{k=1}^{c-1} \left(\!\!\left(\frac{k}{c}\right)\!\!\right)\left(\!\!\left(\frac{ak}{c}\right)\!\!\right)$$

Dedekind sums

$$((x)) = x - \lfloor x \rfloor - \frac{1}{2}$$

Proof: many contour deformations!



Rademacher expansion of the toy example

$$Z = \int_\Gamma \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}\,|\,\eta(\tau)^{24}\,|^2}$$

$$= \frac{(4\pi)^{15}}{24\cdot 13\,!}\sum_{c=1}^{\infty}\;\sum_{a=0,\,(a,c)=1}^{c-1}\frac{\mathrm{e}^{2\pi i\frac{a+a^*}{c}}}{c^2}\left[12ic\,s(a,c)J_{13}(\tfrac{4\pi}{c})+J_{12}(\tfrac{4\pi}{c})-J_{14}(\tfrac{4\pi}{c})\right]+\frac{(4\pi)^{14}\,i}{4\cdot 13\,!}\;.$$

Works for any modular invariant function provided it satisfies a boundedness condition ensuring that the sum converges (analogous to negative weight for the holomorphic Rademacher expansion)

Another example: SO(16) x SO(16) string

One-loop cosmological constant:

$$\Lambda = -\int_{\mathscr{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^6}\frac{1}{\eta(\tau)^{12}}\left(\frac{\vartheta_2^4(\tau)}{\vartheta_2^8(\tilde{\tau})}+\frac{\vartheta_4^4(\tau)}{\vartheta_4^8(\tilde{\tau})}-\frac{\vartheta_3^4(\tau)}{\vartheta_3^8(\tilde{\tau})}\right)$$

$$= \sum_{c\ \mathrm{odd}}\ \sum_{a=0,\,(a,c)=1}^{c-1} \frac{\pi^7 \mathrm{e}^{\frac{\pi i(2a+(c+1)a^*)}{c}}}{240c^2}\left[-8i\sqrt{2}c\, s(a,c)J_5\Big(\frac{2\sqrt{2}\pi}{c}\Big)-J_4\Big(\frac{2\sqrt{2}\pi}{c}\Big)+J_6\Big(\frac{2\sqrt{2}\pi}{c}\Big)\right]$$

Baccianti, Chandra, Hartman, LE, Mizera '25

Closed string four-point function

$$A(s,t) = \sum_{\substack{\text{irreducible} \\ \text{fractions} \\ 0 \leq \frac{a}{c} < 1}} \quad \sum_{\substack{\text{windings} \\ n_\mathrm{L}, n_\mathrm{D}, n_\mathrm{R}, n_\mathrm{U} \in \{0, \ldots, c-1\} \\ n_\mathrm{L} + n_\mathrm{D} + n_\mathrm{R} + n_\mathrm{U} = -a-1 \bmod c}} \quad A_{a/c}^{n_\mathrm{L}, n_\mathrm{D}, n_\mathrm{R}, n_\mathrm{U}}(s,t)$$

Sum over
Ford circles

Additional sum
because of z
Integration

Each contribution
has a Baikov-like
representation with
additional phases

Convergence

- Convergent representation of the amplitude
- Practical for

$$1 \lesssim s \lesssim \mathcal{O}(10 - 50)$$

Convergence gets worse for small *s*

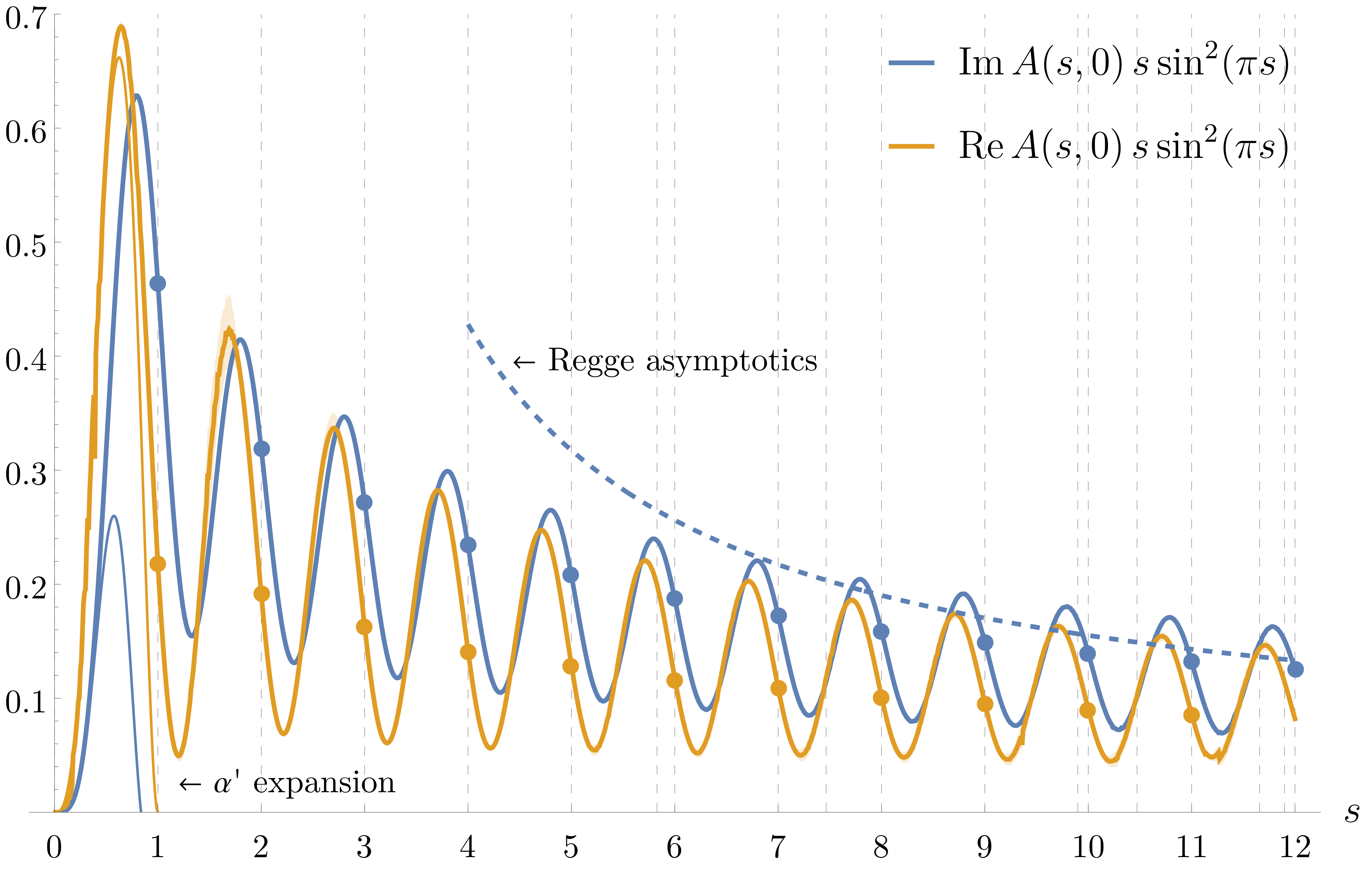
Summing over polarizations becomes hard

- For small *s*: direct numerical integration/α' expansion

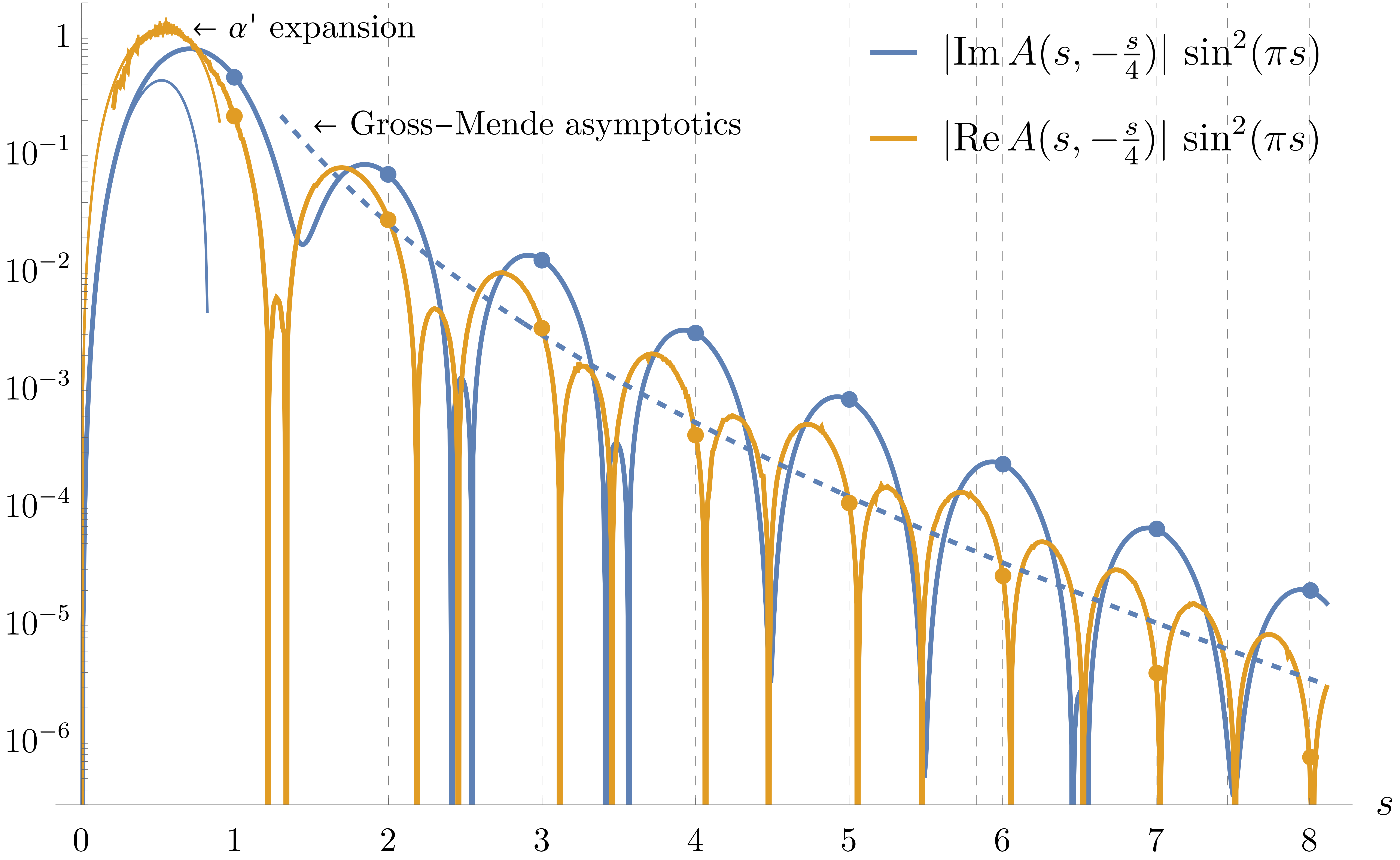
Forward limit (closed string)

- Amplitude mostly imaginary
- Asymptotics understood analytically (error of order $(\log s)^{-1}$)
- consistent with Regge physics

Amati, Ciafaloni, Veneziano '88, Sundborg '88



Fixed angle scattering @ 60 degrees (closed string)



All code available!

III. High-energy limit

The Gross-Mende argument

Gross, Mende '87

- String theory is most interesting at high energies
- At high energies ($s \to \infty$ with scattering angle fixed), amplitudes simplify at every loop order. The integrand has the form

$$\int_{\mathscr{M}_{g,4}} C A^{-s} B^{-t}$$

- So the integral can be done via saddle point approximation:

$$\int_{\mathscr{M}_{g,4}} C A^{-s} B^{-t} \sim \sum_{\text{saddles}} \frac{C_* A_*^{-s} B_*^{-t}}{\sqrt{\det \text{Hess}}}$$

What are all the saddles?

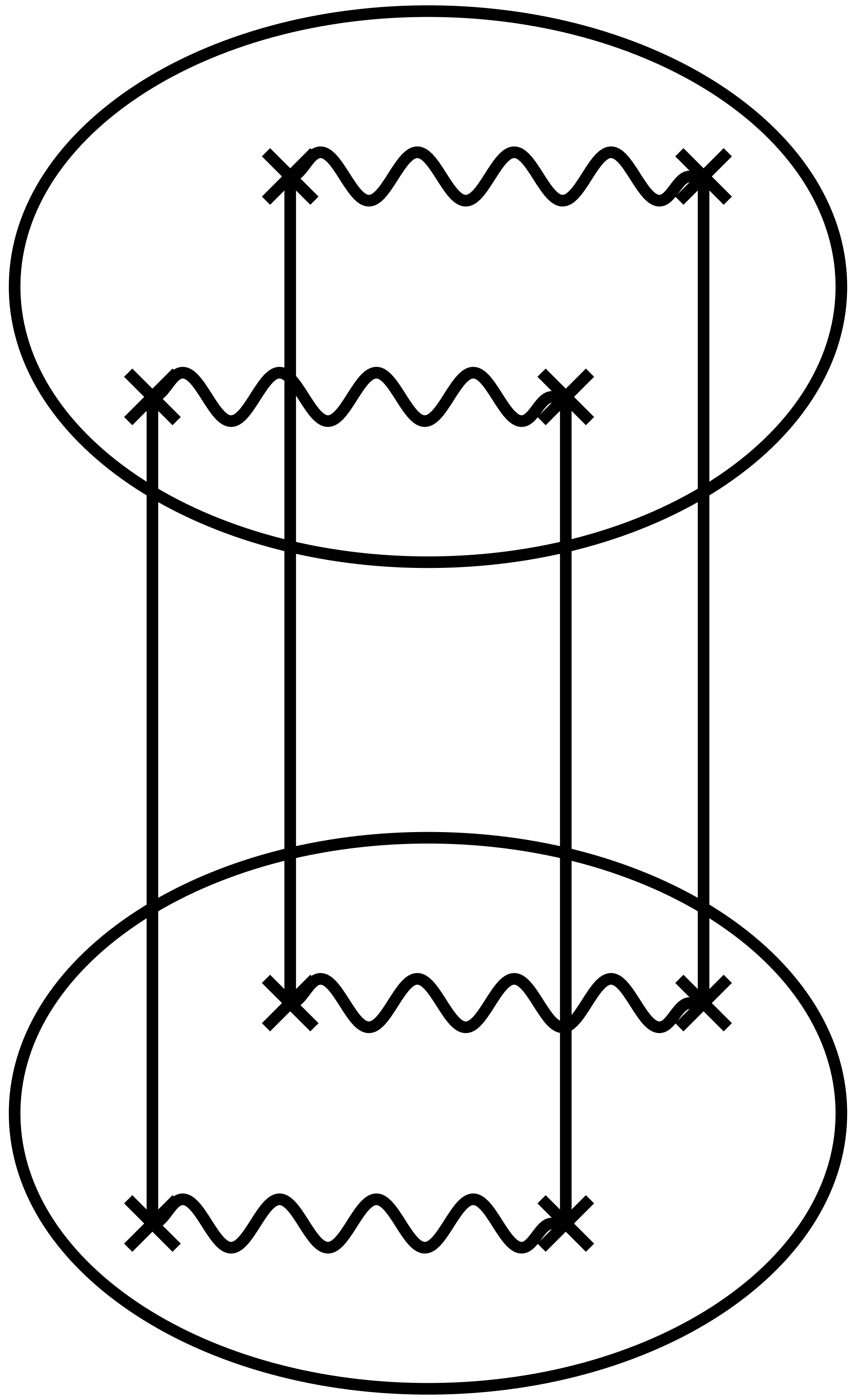
- Tree-level: $\partial_z \log\left(|z|^{-2s}|1-z|^{-2t}\right) = 0 \quad \Longrightarrow \quad z_* = -\frac{s}{u}$

- Gross and Mende identified a set of saddles in $\mathscr{M}_{g,4}$ obtained as a covering surface of the tree-level saddle

$$S_* = \frac{1}{g+1} S_{\text{tree}} = \frac{1}{g+1}\left(s\log(s) + t\log(-t) + u\log(-u)\right)$$

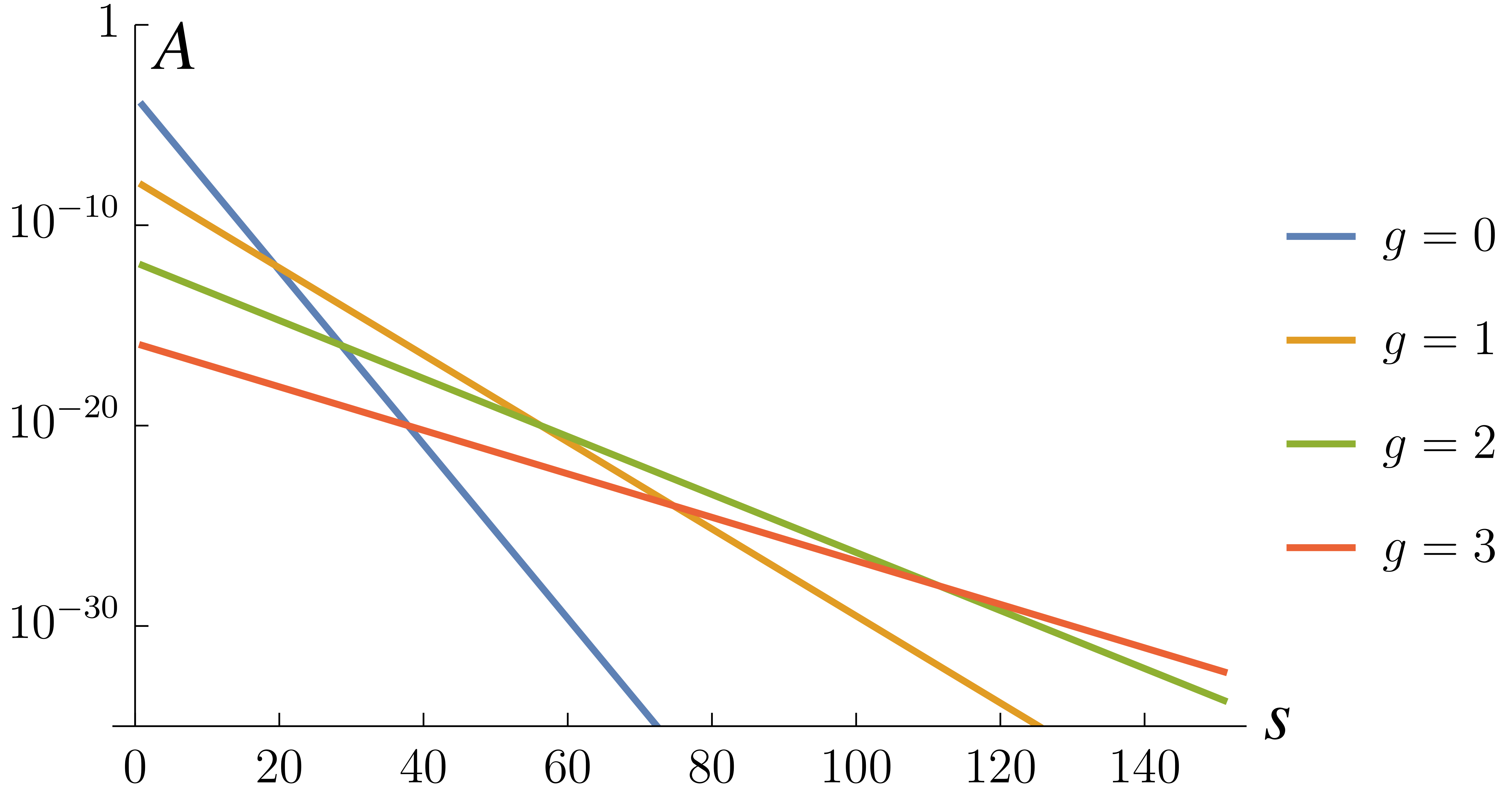
$$A_g \sim s^{1-g} \mathrm{e}^{-\frac{s}{g+1} f(\theta)}$$

- Spacetime perspective: worldsheet looks like a tree-level worldsheet wrapped $g+1$ times around itself



Implications for non-perturbative amplitude

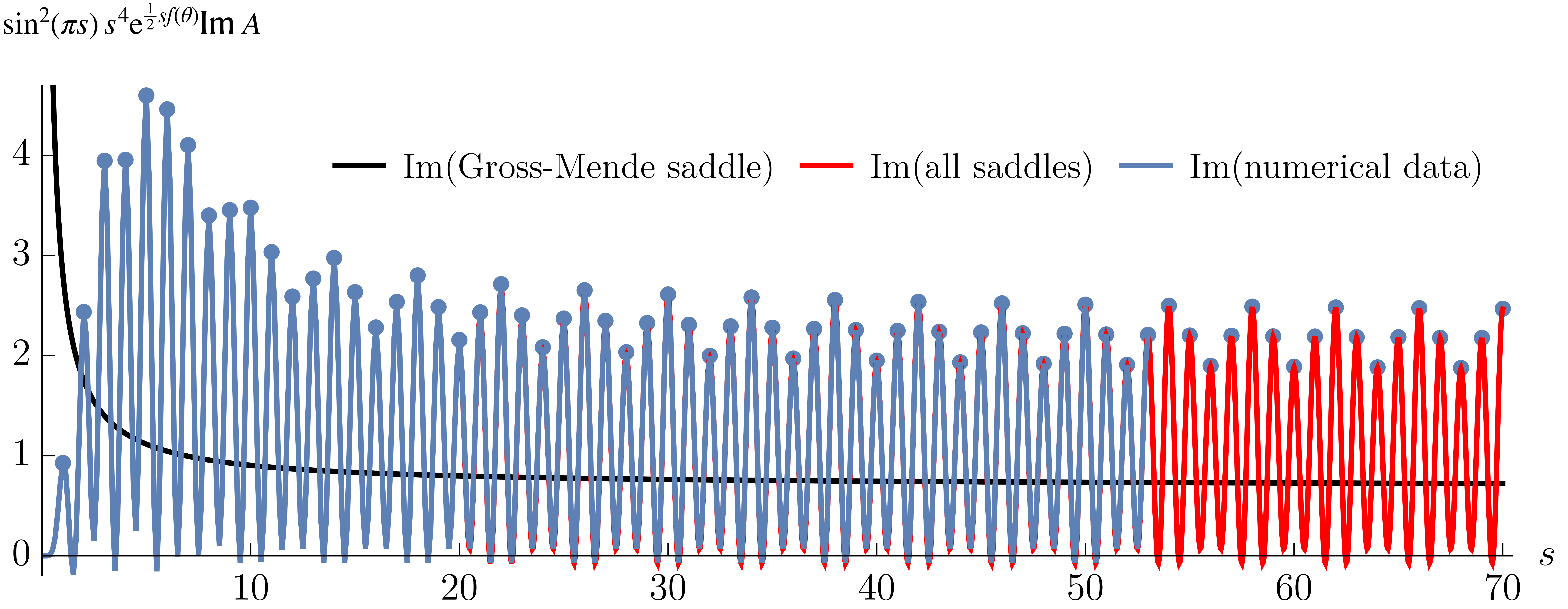
For weak coupling, the non-perturbative amplitude is dominated by a given loop order depending on the energy range



Challenges

- The contour is actually naturally defined in $\mathscr{M}^{\mathbb{C}}_{g,4}$, so we should also search for complex saddles
- Are these all the most dominant saddles?
  - Extensive numerical search at $g = 1$: yes, but
  - There are also saddles with the same $\operatorname{Re} S_*$ not on $\mathscr{M}_{g,4}$: Complex saddles
  - Unknown for $g \geq 2$
- What are their multiplicities?
- Computing the Hessian becomes very hard already at $g = 2$

Let’s test it: imaginary part at 90 degrees



Complex saddles contribute

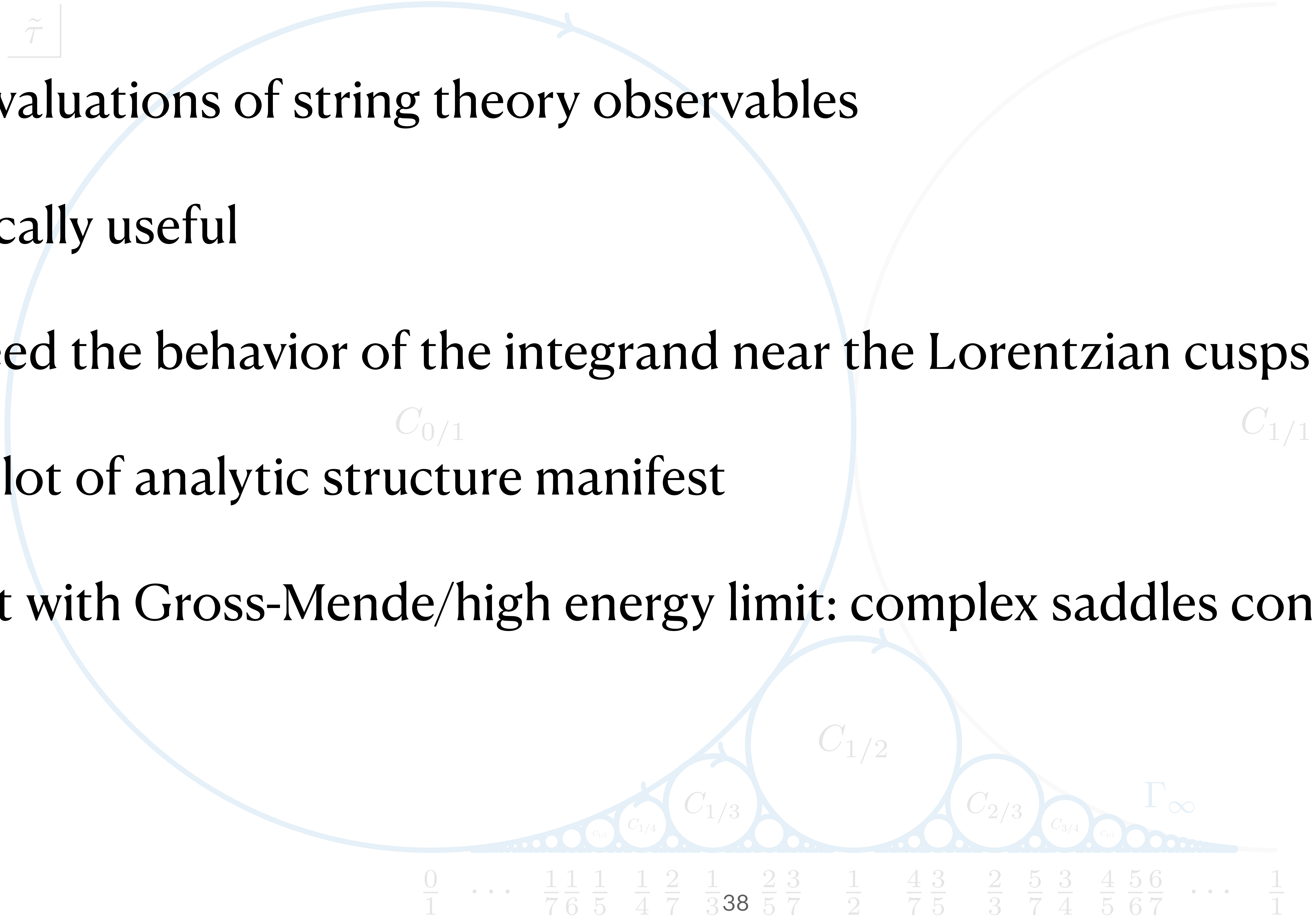
- From the numerical data, we can conclusively see that complex saddles contribute
- Multiplicity of the Gross-Mende saddle $= \dfrac{5\mathrm{e}^{2\pi i s} + 6 + \mathrm{e}^{-2\pi i s}}{4\sin^2(\pi s)}$.
- Complex saddles have $(\tau, \bar{\tau}) = (\gamma \cdot \tau_\theta, \bar{\tau}_\theta)$ with $\gamma \in \Gamma(2)$ instead of them being complex conjugates.
- Saddle multiplicities are determined from consistency conditions + comparing with numerics

Baccianti, LE, Mizera ‘26

Real part is non-zero!

Summary

Contour deformation methods are powerful!



- Exact evaluations of string theory observables
- Numerically useful
- Only need the behavior of the integrand near the Lorentzian cusps
- Make a lot of analytic structure manifest
- Contact with Gross-Mende/high energy limit: complex saddles contribute

Thank you!