The thermal representation of conformal ladder integrals

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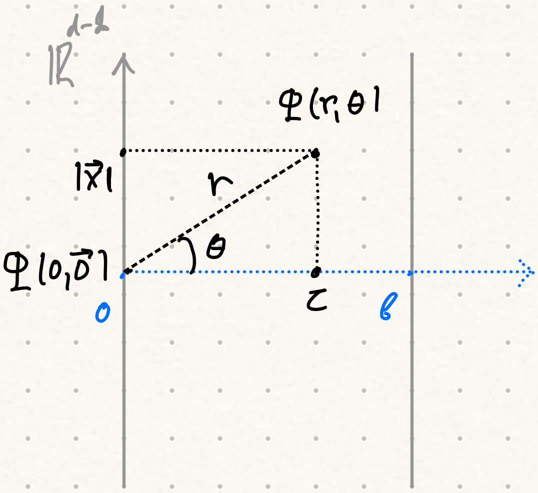
Some context

Thermal 2pt functions in CFTs



$$\Phi(\tau+\beta, \vec{x}) = \Phi(\tau, \vec{x})$$

$S^1_\beta \times \mathbb{R}^{d-1}$

For $\quad 0 < r < \beta$

$$\langle \hat{Q}_s \rangle = \frac{1}{Z_1} \mathrm{Tr}\left[\hat{Q}_s e^{-\beta H}\right]$$

$$\langle \varphi(r,\theta)\varphi(0,0)\rangle = \frac{1}{r^{2\Delta_\varphi}}\left[1 + \sum_{\{Q_s\}} a_{Q_s}\left(\frac{r}{\beta}\right)^{\Delta_{Q_s}} C_s^{d/2-1}(\cos\theta)\right]$$

For massless free fields

$a_{Q_s} \sim J(d-2+s) \qquad s = 2k,\ k \in \mathbb{Z}^+$

i.e. only conserved higher-spin currents contribute.

Question:

How does it evolve?

Is there a smooth deformation to the free values?

$$\underset{(g_i)}{\langle \varphi(r,\theta)\varphi(0,0)\rangle} = \frac{1}{r^{2\Delta_\varphi(g_i)}}\left[1 + \sum_{\{Q_s\}_{(g_i)}} a_{Q_s}(g_i)\left(\frac{r}{\beta}\right)^{\Delta_{Q_s}(g_i)} C_s^{d/2-1}(\cos\theta)\right]$$

"couplings"

Note:

$\Box_x^{(d)}\left(r^s\, C_s^{d/2-1}(\cos\theta)\right) = 0 \implies$ the free 2pt function satisfies: $\Box_x^{(d)} \langle \varphi(x)\varphi(0)\rangle = \delta^{(d)}(x)$

The integrable deformation:

$$S_0 = \int_0^\beta d\tau \int d^{d-1}x \, \frac{1}{2} \partial^\mu \varphi \, \partial_\mu \varphi$$

$$S = \int_0^\beta d\tau \int d^{d-1}x \left[|D_\mu \varphi|^2 + m^2 |\varphi|^2 \right] , \quad \varphi(\tau+\beta, \vec{x}) = e^{i\mu\beta} \varphi(\tau, \vec{x})$$

$$D_\mu = \partial_\mu - i A_\mu , \quad A_\mu = (\mu, \vec{0})$$

$$\hat{H} = \hat{H}_0 + m^2 \hat{O} + i\mu \hat{Q} \quad \Big\| \quad \hat{O} = |\varphi|^2 , \quad \hat{Q} = \varphi^* \overleftrightarrow{\partial_\tau} \varphi$$

We obtain:

$$\langle \varphi^*(r,\theta) \varphi(0,0) \rangle = \frac{1}{r^{2\Delta_\varphi}} \left[1 + \sum_{\{Q_s\}} a_{Q_s}(z,\bar{z}) \left(\frac{r}{\beta} \right)^{\Delta_{Q_s}} C_s^{d/2-1}(\cos\theta) + \text{"shadows"} \right]$$

The spectrum is

$\Delta_{Q_s} = d-2+s+2t$

$s, t \in \mathbb{Z}^+$

$z = e^{-\beta m - i\beta\mu}$

Scalars with

$\Delta_{\sigma^k} = 2k , \quad k = 1, 2, \ldots$

Note:

_ The 2pt function is no longer a Green's function of the Laplacian, due to higher twists and shadows.

– It is CFT-like, despite being massive.

The result for the higher-spin contributions:

$$a^L_{Q_s}(z,\bar{z}) = \frac{\Gamma(L-1/2)}{\Gamma(L+s-1/2)(4\pi)^L 2^{2s}} \sum_{n=0}^{L-1+s} \frac{2^n \ln^n|z|\,(2L-2+s-n)!}{n!\,(L-1+s-n)!} \times$$

$$d = 2L+1 \qquad \times \left[Li_{2L-1+s-n}(z) + (-1)^s Li_{2L-1+s-n}(\bar{z}) \right]$$

To complete the context, we calculate the thermal partition function $\longrightarrow$ free energy.

$$Z_L = \int D\varphi\, e^{-S}$$

$$\ln Z_L(z,\bar{z}) = \frac{(-1)^L L!\,(2\ln|z|)^{2L+1}}{2(2L+1)!} + \sum_{n=0}^{L} \frac{(2L-n)!\,(-2\ln|z|)^n}{n!\,(L-n)!} 2\,\mathrm{Re}\left[Li_{2L+1-n}(z)\right]$$

This means that one can calculate thermal 1pt functions taking appropriate derivatives of the above.

We can do that for $a^L_{Q_0} \sim \langle|\varphi|^2\rangle$ and $a^L_{Q_L} \sim \langle \varphi^\dagger \overleftrightarrow{D}_\tau \varphi \rangle$

... but let us introduce some notation to help us with the global emerging picture...

Notations

A simple but apparently overlooked property of the free energies $\ln Z_L(z,\bar{z})$ is the following:

The parent quantum mechanical model.

Consider the Hamiltonian.

$$\hat{H} = \frac{1}{2}\hat{p}_1^2 + \frac{1}{2}\hat{p}_2^2 + m^2 \frac{1}{2}(\hat{x}_1^2+\hat{x}_2^2) + i\mu(\hat{p}_1\hat{x}_2 - \hat{p}_2\hat{x}_2)$$

$$= \hat{H}_0 + m^2\hat{O} + i\mu\hat{Q}$$

Using an appropriate set of creation/anihilation operators

$$\hat{H}_0 + m^2\hat{O} = m(\hat{N}_1+\hat{N}_2+1)$$
$$\hat{Q} = \hat{N}_1 - \hat{N}_2$$

$: \hat{N}_i = \hat{a}_i^+ a_i, \; i=1,2.$

We calculate the thermal partition function

$$Z_0 = \mathrm{Tr}\, e^{-\beta(\hat{H}_0 + m^2\hat{O} + i\mu\hat{Q})}$$

$$= \sum_{n_1,n_2=0}^{\infty} e^{-\beta m(n_1+n_2+1) - i\beta\mu(n_1-n_2)}$$

$$= e^{\ln|z| - \ln(1-z) - \ln(1-\bar{z})}$$

$$\Rightarrow \boxed{\ln Z_0(z,\bar{z}) = \ln|z| - \ln(1-z) - \ln(1-\bar{z})}$$

Next define:

$$\hat{D}_z = \frac{1}{\beta^2}\frac{\partial}{\partial m^2} = \frac{1}{2\ln|z|}\left(z\partial_z + \bar{z}\partial_{\bar{z}}\right)$$

$$\hat{L}_z = \frac{i}{\beta}\frac{\partial}{\partial \mu} = z\partial_z - \bar{z}\partial_{\bar{z}}$$

to obtain

$$\langle \hat{O}(z,\bar{z})\rangle_0 = -\beta\, \hat{D}_z * \ln Z_0(z,\bar{z}) = \frac{1}{2m}\,\frac{1-|z|^2}{|1-z|^2}$$

$$\langle \hat{Q}(z,\bar{z})\rangle_0 = \hat{L}_z * \ln Z_0(z,\bar{z}) = \frac{z-\bar{z}}{|1-z|^2}$$

Note:

These are harmonic functions on the open disk and the upper half plane, taking constant values at the boundary

From $\ln Z_0$ to $\ln Z_L$

Remarkably, using the relativistic one-particle density of states

$$\rho_L(\omega; m; \alpha^2) = \frac{2\alpha^{2L}\beta^{2L}}{(L-1)!}\,\omega\,(\omega^2 - m^2)^{L-1}$$

we obtain:

$$\ln Z_L(z,\bar{z};\alpha^2) = \int_m^\infty d\omega\, \rho_L(\omega;m;\alpha^2)\, \ln Z_0(z',\bar{z}')$$

$$= -\frac{\alpha^{2L}}{(L-1)!}\int_0^{|z|} \frac{d|z'|}{|z'|}\, 2\ln|z'|\left(\ln^2|z'| - \ln^2|z|\right)^{L-1} \ln Z_0(z',\bar{z}')$$

$$\omega^2 = m^2 + \vec{p}^{\,2}, \quad V_{d-1} = \ell^{d-1}, \quad \vec{p} = \frac{2\pi}{\ell}(n_1, \dots, n_{d-1})$$

$$\alpha^2 = \frac{e^2}{4\pi b^2}$$

The precise result is:

$$\ln Z_L(z,\bar{z};\alpha^2) = \alpha^{2L} \ln Z_L(z,\bar{z})$$

A bit more notation

The above formulae can be recast as iterated integrals

$$\ln Z_L(z,\bar{z}) = (-1)^L \left\{ \mathrm{ord} \prod_{i=1}^{L} \right\} \left[\int_0^{|z_{i+1}|} \frac{d|z_i|}{|z_i|}\, 2\ln|z_i| \right] \ln Z_0(z_1,\bar{z}_1)$$

the integrals are ordered as $0 \le |z_1| \le |z_2| \cdots \le |z_L| \le |z|$

Define the integral operator

$$\boxed{\check{d}_{z;z'} = \int_0^{|z|} \frac{d|z'|}{|z'|} \, 2\ln|z'|}$$

such that

$$\boxed{\begin{aligned} \ln Z_L(z,\bar{z}) &= (-1)^L \check{d}_{z;z_L} * \cdots * \check{d}_{z_2;z_1} * \ln Z_0(z_1,\bar{z}_1) \\ &= [-\check{d}]^L_{z;z_1} * \ln Z_0(z_1,\bar{z}_1) \end{aligned}}$$

There is yet another set of functions that we can obtain from $\ln Z_0(z,\bar{z})$.

$$\boxed{\begin{aligned} (z-\bar{z})^k \left(\frac{1}{z-\bar{z}} \hat{L}_z \right)^k * \ln Z_0(z,\bar{z}) &= \Gamma(k) \frac{(z-\bar{z})^k}{|1-z|^{2k}} \\ k \in \mathbb{Z}^+ \qquad\qquad &\equiv \mathcal{D}_0^k(z,\bar{z}) \end{aligned}}$$

The parameter k is the depth. It is raised as

$$\boxed{\hat{L}_z^{(k)} * \mathcal{D}_0^k(z,\bar{z}) = \mathcal{D}_0^{k+1}(z,\bar{z})}$$

with

$$\boxed{\hat{L}_z^{(k)} = \hat{L}_z - k\,\frac{z+\bar{z}}{z-\bar{z}}}$$

Summary of notations

We have defined differential and integral operators that act on $\ln Z_0(z,\bar{z})$

$$\ln Z_0(z,\bar{z}) \equiv \mathcal{D}_0^0(z,\bar{z}) = \ln|z| - \ln(1-z) - \ln(1-\bar{z})$$

$$\ln Z_L(z,\bar{z}) \equiv \mathcal{D}_L^0(z,\bar{z})$$

$$\Gamma(k)\frac{(z-\bar{z})^k}{|1-z|^{2k}} \equiv \mathcal{D}_0^k(z,\bar{z}) = \left[\hat{L}_z\right]^k * \ln Z_0(z,\bar{z})$$

$$[\hat{D}_z, \hat{L}_z] = 0 \,, \quad [\hat{L}_z^{(k)}, \hat{D}_z] = 0$$

$$[\check{d}_{z;z'}, \hat{D}_{z'}] = 0 \,, \quad [\hat{L}_z, \check{d}_{z;z'}] = 0$$

$$\ln Z_L(z,\bar{z}) = [-\check{d}]^L_{z;z'} * \ln Z_0(z',\bar{z}')$$

$$\hat{D}_z * \ln Z_L(z,\bar{z}) = \ln Z_{L-1}(z,\bar{z}) = -\frac{1}{6}\langle \hat{O}(z,\bar{z})\rangle_L$$

$$\hat{L}_z * \ln Z_L(z,\bar{z}) = \langle Q(z,\bar{z})\rangle_L$$

$$\check{d}_{z;z'} * \ln Z_L(z',\bar{z}') = -\ln Z_{L+1}(z,\bar{z})$$

Conformal ladder integrals

The following class of L-loop 4pt integrals in $D=2k+2$ dims.

are omnipresent in CFT calculations

$$g^{2L} I_L^k(x_1,\dots,x_4) = \prod_{n=1}^{L}\left[\int \frac{d^{2k+2}y_n}{\pi^{k+1}}\, \frac{g^2\,\Gamma(k)}{|y_{n-1,n}|^2\,|x_2-y_n|^2\,|x_4-y_n|^2}\right]\frac{1}{|y_L-x_3|^{2k}}$$

$y_{ij}=y_i-y_j$, $y_0=x_1$, $g^2 \sim$ loop counting parameter

From conformal invariance we have:

$$I_L^k(x_1,\dots x_4) = \frac{1}{x_{13}^{2k}\,x_{24}^{2L}}\,\Phi_L^k(u,v)$$

$$u=\frac{x_{12}^2\,x_{34}^2}{x_{13}^2\,x_{24}^2}\ , \quad v=\frac{x_{12}^2\,x_{34}^2}{x_{14}^2\,x_{23}^2}$$

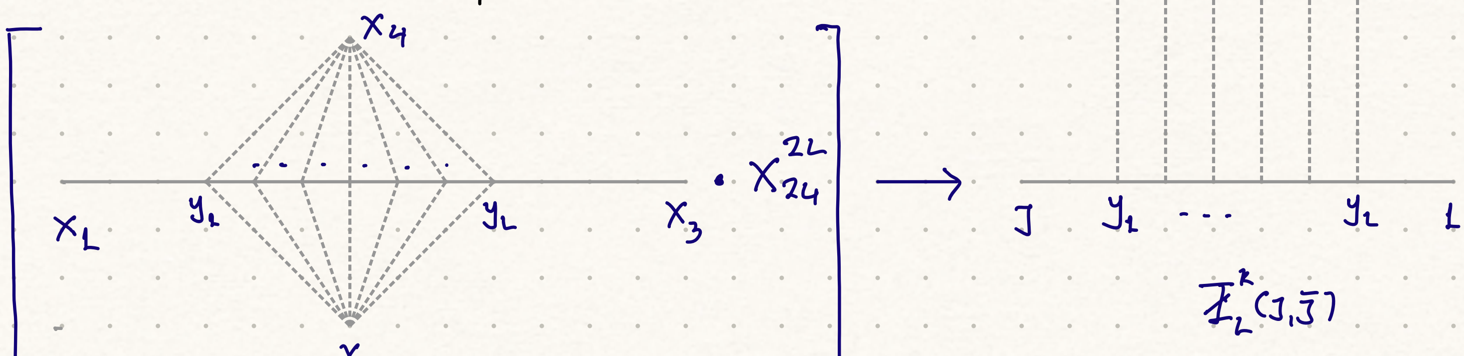
and

$$\lim_{\{x_i\}\to\{z,\infty,1,0\}}\left[x_{24}^{2L}\, I_L^k(x_1,\dots x_4)\right] = \mathcal{I}_L^k(z,\bar z) = \frac{1}{|1-z|^{2k}}\,\Phi_L^k(z,\bar z)$$

In this limit

$$(u,v)\longrightarrow\left(\frac{1}{(1-z)(1-\bar z)}\,,\ \frac{1}{z\bar z}\right)$$

Schematically



$$x_2$$

$$x_1 \to z$$
$$x_2 \to \infty$$
$$x_3 \to 1$$
$$x_4 \to 0$$

$$z_1 \text{———} z_2 = \frac{1}{|z_1 - z_2|^{2k}}, \qquad z_1 \text{- - - -} z_2 = \frac{1}{|z_1 - z_2|^{2}}$$

A remarkable result by Isaev, using <u>conformal quantum mechanics</u> yields:

$$I_L^k(J,\bar{J}) = \frac{1}{L!\,(L-1)!}\int_0^\infty dt\, 2(t - \ln|J|)\left[t(t - 2\ln|J|)\right]^{L-1} \frac{e^{-kt}}{|1 - J e^{-t}|^{2k}}$$

Using the change of variables

$$J' = J e^{-t}, \quad J = |J| e^{i\theta}, \quad J' = |J'| e^{i\theta}$$

we find

$$\boxed{\Phi_L^k(J,\bar{J}) = \frac{1}{D_0^k(J,\bar{J})\, L!\,(L-1)!}\int_0^{|J|} \frac{d|J'|}{|J'|}\, 2\ln|J| \left(\ln^2|J'| - \ln^2|J|\right)^{L-1} D_0^k(J',\bar{J}')}$$

The correspondence

Setting

$$J \longleftrightarrow z, \quad g^2 \longleftrightarrow \alpha^2$$

we can map the thermal free energies to conformal ladder graphs as:

$$\boxed{\mathcal{I}^k_L(\eta,\bar\eta) = \frac{1}{|1-\eta|^{2k}}\,\Phi^k_L(\eta,\bar\eta) \longleftrightarrow \frac{1}{\Gamma(k)\,L!}\,\frac{1}{(z-\bar z)^k}\,\left[-\check{d}_{z;z'}\right]^L * \left[\hat{L}_{z'}\right]^k * \ln Z_0(z',\bar z') \\ = \frac{1}{\Gamma(k)\,L!}\,\frac{1}{(z-\bar z)^k}\,\mathcal{D}^k_L(z,\bar z)}$$

An inviting form of the above is:

$$\boxed{\Phi^k_L(z,\bar z) = \frac{1}{L!}\,\frac{\mathcal{D}^k_L(z,\bar z)}{\mathcal{D}^k_0(z,\bar z)}}$$

Schematically

$$k = \frac{D-2}{2} \qquad L = \frac{d-1}{2}$$

$$\begin{tikzcd}
 & 0 & 1 & 2 & 3 \\
0 & \ln Z_0 \arrow[r, "\hat{\mathbf{L}}_z^{(0)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^1_0 \arrow[r, "\hat{\mathbf{L}}_z^{(1)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^2_0 \arrow[r, "\hat{\mathbf{L}}_z^{(2)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^3_0 \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] \\
1 & \mathcal{D}^0_1 \arrow[r, "\hat{\mathbf{L}}_z^{(0)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^1_1 \arrow[r, "\hat{\mathbf{L}}_z^{(1)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^2_1 \arrow[r, "\hat{\mathbf{L}}_z^{(2)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^3_1 \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] \\
2 & \mathcal{D}^0_2 \arrow[r, "\hat{\mathbf{L}}_z^{(0)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^1_2 \arrow[r, "\hat{\mathbf{L}}_z^{(1)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^2_2 \arrow[r, "\hat{\mathbf{L}}_z^{(2)}"] \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] & \mathcal{D}^3_2 \arrow[d, "-\check{\mathbf{d}}_{z;z'}"'] \\
3 & \mathcal{D}^0_3 \arrow[r, "\hat{\mathbf{L}}_z^{(0)}"] & \mathcal{D}^1_3 \arrow[r, "\hat{\mathbf{L}}_z^{(1)}"] & \mathcal{D}^2_3 \arrow[r, "\hat{\mathbf{L}}_z^{(2)}"] & \mathcal{D}^3_3
\end{tikzcd}$$

The differential equation

The functions $\mathcal{D}_L^k(z,\bar{z})$ satisfy a familiar-looking differential equation $\longrightarrow$ after some calculations:

$$\boxed{\begin{gathered}\left(\frac{1}{4b^2}\hat{\Delta}_z - L\hat{D}_z + k(k-1)\frac{|z|^2}{(z-\bar{z})^k}\right) * \mathcal{D}_L^k(z,\bar{z}) = 0 \\ \hat{\Delta}_z = 4b^2 z\bar{z}\,\partial_z\partial_{\bar{z}}\end{gathered}}$$

In terms of the variables m, μ:

$$\boxed{\left(\frac{\partial^2}{\partial m^2} + \frac{\partial^2}{\partial \mu^2} - 2L\frac{1}{m}\frac{\partial}{\partial m} - b^2\frac{k(k-1)}{\sin^2(b\mu)}\right)\mathcal{D}_L^k(m,\mu) = 0}$$

For $L=0$ ($d=1$, quantum mechanics), this coincides with the e.o.m. of a scalar field in Euclidean AdS_2 with metric

$$ds^2 = \frac{1}{\sin^2(b\mu)}(dm^2 + d\mu^2)$$

and mass

$$\mathcal{M}^2 = b^2 k(k-1)$$

Hence, $\mathcal{D}_0^k(m,\mu)$ are bulk-to-boundary propagators in the context of AdS_2/CFT_1, for operators at the bdy of AdS_2 – at $m=0$ ($|z|=1$) – with scaling dimension k.

Recap

Conformal ladder integrals		Thermal one-point functions
Dimension D	$D = 2k+2$	Depth k
Loop order L	$d = 2L+1$	Dimension d
Spacetime points $x_i = (0, 1, z, \infty)$	$z = e^{-\beta m - i\beta\mu}$	Mass m, chemical potential μ
Dimensionless coupling g^2	$g^2 = \alpha^2$	Geometric parameter $\alpha^2 = \dfrac{l^2}{4\pi\beta^2}$

Back to thermal 2pt functions

It appears that we have only managed to evaluate the spin-zero and spin-one thermal 1pt functions.

$$\boxed{\begin{aligned} a^L_{Q_0}(z,\bar{z}) &= \frac{L}{(4\pi)^L \alpha^{2L}} \mathcal{D}^0_{L-L}(z,\bar{z}) \\ a^L_{Q_L}(z,\bar{z}) &= \frac{L}{(4\pi)^L \alpha^{2L}} \frac{1}{2} \mathcal{D}^L_L(z,\bar{z}) \end{aligned}}$$

What about the 1pt functions of higher-spin currents?

Note:

The massive theories we consider do not have conformal HS currents!

However, the thermal expansion requires the presence of HS conformal currents!

So, where $a^L_{Q_S}$ correspond to?

In the free theories we can construct explicitly the symmetric, traceless and conserved conformal HS currents that give rise to the 1pt functions $a^L_{Q_s}$.

$$\boxed{Q^s_{\mu(s)} = \bar{\Phi}^*(\partial - iA)_{\mu_1} \cdots (\partial - iA)_{\mu_s}\Phi + (-1)^s\, \text{h.c.} - \text{traces}}$$

with

$$\langle Q^s_{\mu(s)} \rangle_L = \frac{b^L_{Q_s}}{\ell^{\Delta_{Q_s}}} \left(e_{\mu_1} \cdots e_{\mu_s} - \text{traces}\right)$$

$$a^L_{Q_s} = \frac{f^L_{\varphi\varphi Q_s}\, b^L_{Q_s}}{C^L_{Q_s}} \frac{s!\, \Gamma\left(\frac{d-2}{2}\right)}{2^s\, \Gamma\left(\frac{d-2}{2}+s\right)} = b^L_{Q_s} \frac{\Gamma\left(\frac{d-2}{2}\right)}{2^s\, \Gamma\left(\frac{d-2}{2}+s\right)}$$

Next, a tedious calculation gives the recursive relation

$$\boxed{b^d_{Q_{s+2}} = 2\pi(d+2s)\, b^{d+2}_{Q_s} + (\ell m)^2\, b^d_{Q_s}}$$

or equivalently

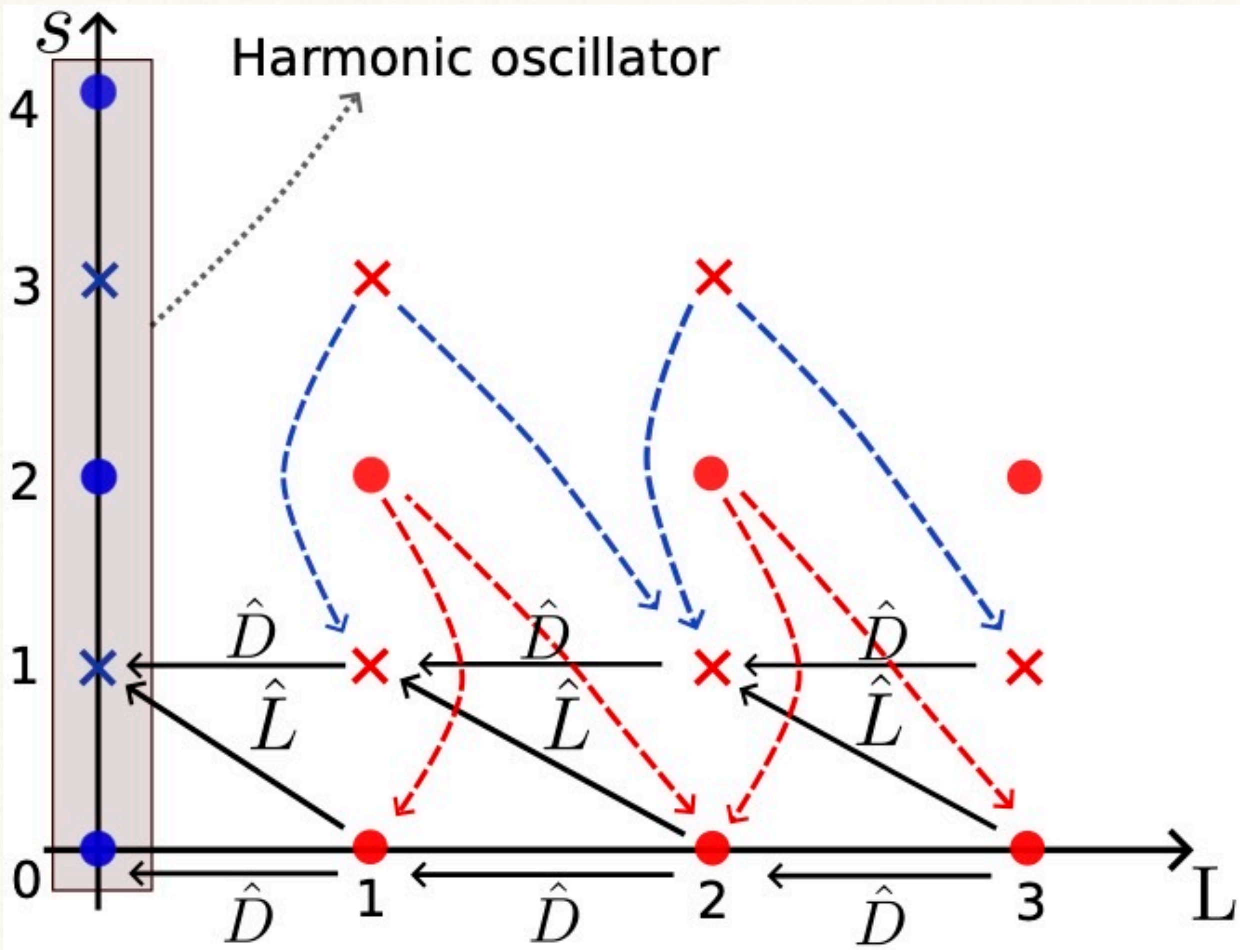
$$\boxed{a^L_{Q_{s+2}}(z,\bar{z}) = \frac{2\pi}{2L-1}\, a^{L+1}_{Q_s}(z,\bar{z}) + \frac{\operatorname{li}^2(z)}{(2L-1+2s)(2L+1+2s)}\, a^L_{Q_s}(z,\bar{z})}$$

The derivation is possible due to the equation of motion

$$\boxed{(\partial - iA)^2\Phi = m^2\Phi}$$

The result shows that higher-spin operators are determined by lower-spin operators in higher dimensions.

Eventually, the thermal 1pt function of a higher-spin operator is given as a sum of scalar (or spin-one) thermal 1pt functions in higher-dimensions.



The hyper-partition function and the all-loop resummation

Let us define the hyper-partition function and the corresponding hyper-free energy as:

$$Z(z,\bar{z};\alpha^2) = \prod_{L=0} Z_L(z,\bar{z};\alpha^2)$$

$$\ln Z(z,\bar{z};\alpha^2) = \sum_{L=0}^{\infty} \ln Z_L(z,\bar{z};\alpha^2)$$

The latter satisfies:

$$\left(\frac{\partial}{\partial m} + 2\alpha^2 b^2 m\right) \ln Z(z,\bar{z};\alpha^2) = -b\,\frac{\sinh(bm)}{\cosh(bm) - \cos(b\mu)}$$

This can be integrated as:

divergent

$$\ln Z_L(m,\mu;\alpha^2) = \ln Z_0(m,\mu) + e^{-\alpha^2 b^2 m^2}\left[e^{\alpha^2 b^2 \omega^2} \ln Z_0(\omega,\mu)\right]_m^{\infty}$$
$$+\, b\, e^{-\alpha^2 b^2 m^2} \int_m^{\infty} d\omega\, e^{\alpha^2 b^2 \omega^2} \frac{\sinh(b\omega)}{\cosh(b\omega) - \cos(b\mu)}$$

The analytic continuation $\alpha \to i\alpha$ yield a controlable result and it is relevant for the resummation of ladder integrals

$$\boxed{I^k(z,\bar{z};g^2) = \sum_{L=0}^{\infty} (-g^2)^L I_L^k(z,\bar{z}) = \frac{1}{\Gamma(k)(z-\bar{z})^k}\left[\hat{L}_z\right]^k_* \sum_{L=0}^{\infty} \frac{1}{L!} \ln Z_L(z,\bar{z};-\alpha^2)}$$

Hence we need the Borel sum of $\ln Z_L$:

$$\sum_{L=0}^{\infty} \frac{1}{L!} \ln Z_L(m,\mu;\alpha^2) = \ln Z_0(m,\mu) + \alpha b \int_m^{\infty} 2\omega\, d\omega\, \frac{I_1\left[2\alpha b\sqrt{\omega^2 - m^2}\right]}{\sqrt{\omega^2 - m^2}} \ln Z_0(m,\mu)$$

The replacement $\alpha \longrightarrow ig$ gives

$$\boxed{\sum_{L=0}^{\infty} \frac{1}{L!} \ln Z_L(m,\mu;-g^2) = \ln Z_0(z,\bar{z}) + \left[J_0\left(2gb\sqrt{\omega^2-m^2}\right) \ln Z_0(\omega,\mu) \right]_m^{\infty} + b\int_m^{\infty} d\omega \, J_0\left(2gb\sqrt{\omega^2-m^2}\right) \frac{\sinh(b\omega)}{\cosh(b\omega)-\cos(b\mu)}}$$

Now the divergence comes only from the vacuum energy in $\ln Z_0(\omega,\mu)$.

If we want to obtain the result for $\underline{D=4 \Rightarrow k=1}$ and we apply $\hat{L}_z$ just once. We get:

$$\boxed{I^1(z,\bar{z};g^2) = \frac{1}{2|z|} \int_m^{\infty} d\omega \, J_0\left(2gb\sqrt{\omega^2-m^2}\right) \frac{\sinh(b\omega)}{\left(\cosh(b\omega)-\cos(b\mu)\right)^2}}$$

a result first obtained by Broadhurst and Davydychev (2010)

Outlook

- AGT-like result: the conformal cross ratios $J, \bar{J}$ are identified with moduli (i.e. instanton moduli) $z, \bar{z}$.
- The Borel sum of free energies is reminiscent of a genus expansion.
- Consider other deformations of the massless free

theories e.g. $g\varphi \longrightarrow$ e.o.m $\Box\varphi = g\varphi$.

- Does the KMS condition of thermal 2pt functions affect the thermal 1pt functions?

- The second order differential equation obeyed by $\mathcal{D}_L^{\kappa}(z,\bar{z})$ is closely related to differential equations obeyed by modular forms. This connects thermal 1pt functions to string amplitudes.

- Integrability studies: our $\hat{D}_z$ & $\hat{L}_z$ operators have been used to show a Toda-like structure of ladder integrals and also the antipodal symmetry of conformal ladder graphs.

- What about conformal invariance of the thermal theory? For general (m, μ), this is not a thermal CFT. Conformality is determined by the gap equation.

$$\langle \varphi(\tau,\vec{0})\,\varphi(0,\vec{0})\rangle = \frac{1}{\tau^{2\Delta_\varphi}}\left[1 + a^{2}_{\varphi^2}\left(\frac{\tau}{\beta}\right)^{d-2} + \cdots\right] \qquad \Delta_\varphi = d/2 - 1$$

$$= \frac{1}{\beta^{d-2}}\, a_{\varphi^2} + \frac{1}{\tau^{2\Delta_\varphi}}\sum_{Q_s}\tilde{a}_{Q_s}\left(\frac{\tau}{\beta}\right)^{\Delta_{Q_s}}$$

It turns out (i.e. in large-N vector models) that the operator φ^2 drops out from the spectrum at the non-trivial critical point. This gives an algebraic equation for βm (assume e.g. $\mu = 0$).

$L=1,\ d=3:\qquad \beta m + \ln(1-e^{-\beta m}) = 0$

$L=2,\ d=5\qquad \frac{1}{6}(\beta m)^3 + Li_3(e^{-\beta m}) + \beta m\, Li_2(e^{-\beta m}) = 0$

$\vdots \qquad \vdots$

We learn that for $d=3$ the non-trivial CFT is stable (real solution for βm), while for $d=5$ is unstable (complex solutions).